Graph Theory: CMSC 27530/37530 Lecture 15

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Revised by instructor

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HW+ and **Bonus+** indicate homework and Bonus problems due a week from the date the problem was assigned in class; in this case, due next Tuesday.

GIRTH OF A GRAPH

Definition 15.1. The **girth** of a graph G, written girth(G), is the length of the shortest cycle.

Examples 15.2.

1. girth$(K_n) = 3$.
2. girth$(K_{r,s}) = 4$.
3. girth(Petersen) $= 5$.
4. girth(tree) $= \infty$.

HW 15.3. (5+2 points)

(a) If G is r-regular and girth$(G) \geq 5$, then $n \geq r^2 + 1$.

(b) Show equality can hold for $r = 1, 2, 3$. "Can hold" means there exists an r-regular graph for which $n = r^2 + 1$.

Remark 15.4. Equality can also hold for $r = 7$, but don't try to show this.
Later we will prove that if equality holds, then $r \in \{1, 2, 3, 7, 57\}$.

LAPLACIAN, THE MATRIX–TREE THEOREM

Definition 15.5. If G is a graph and A_G is the adjacency matrix, then the **Laplacian** of G is the matrix L_G defined by

$$L_G = \text{diag}(\deg(1), \ldots, \deg(n)) - A_G.$$

Note that L_G is a symmetric matrix.

Notation 15.6. We denote the vector $(1, 1, \ldots, 1)^T$ by $\mathbf{1}$.

Observe that each row sum of the Laplacian is zero and therefore

$$L_G \mathbf{1} = \mathbf{0}. \tag{1}$$

DO 15.7. Infer from Eq. (1) that $\det(L_G) = 0$.

Theorem 15.8 (Matrix–Tree Theorem, Kirchhoff 1848)**.**

$$\det({}_{\widehat{i}}(L_G)_{\widehat{i}}) = \#\text{ of spanning trees.}$$

Here, matrix ${}_{\widehat{i}}(L_G)_{\widehat{i}}$ *is the* $(n-1)\times(n-1)$ *matrix obtained by deleting the* i*-th row and the* i*-th column of* L_G*.*

HW 15.9. (5 points) Use Theorem 15.8 to give another proof of Cayley's formula:

$$\#\text{ of spanning trees of } K_n = n^{n-2}.$$

Definition 15.10. A symmetric matrix $A \in M_n(\mathbb{R})$ is called **positive semidefinite** if $(\forall \mathbf{x} \in \mathbb{R}^n)(\mathbf{x}^T A \mathbf{x} \geq 0)$.

HW 15.11. (5 points) Prove: A symmetric matrix $A \in M_n(\mathbb{R})$ is positive semidefinite if and only if all its eigenvalues are non-negative.

BONUS+ 15.12. (6 points) Prove: L_G is *positive semidefinite.*

INDEPENDENCE OF EVENTS AND OF RANDOM VARIABLES

Let (Ω, P) be a finite probability space.

Definition 15.13. Two events $A, B \subseteq \Omega$ are independent if $P(A \cap B) = P(A) \cdot P(B)$.

Definition 15.14. k events, $A_1, \ldots, A_k$, are independent if

$$(\forall I \subseteq [k]) \left(P\left(\bigcap_{i \in I} A_i \right) = \prod_{i \in I} P(A_i) \right).$$

Definition 15.15. Random variables $X_1, \ldots, X_k : \Omega \to \mathbb{R}$ are independent if

$$(\forall \alpha_1, \ldots, \alpha_k \in \mathbb{R})(P(X_1 = \alpha_1 \wedge \cdots \wedge X_k = \alpha_k) = \prod_{i=1}^{k} P(X_i = \alpha_i)).$$

DO 15.16. Events $A_1, \ldots, A_k$ are independent $\iff$ their indicator random variables are independent.

DO 15.17. If $A_1, \ldots, A_k$ are independent, then $A_1, \ldots, A_{k-1}, \overline{A_k}$ are independent.

Corollary 15.18. *If* $A_1, \ldots, A_k$ *are independent, then* $A_1^{\epsilon_1}, \ldots, A_k^{\epsilon_k}$ *are independent* $\forall \epsilon_i \in \{\pm 1\}$*, where*

$$A_i^{\epsilon_i} = \begin{cases} A_i & \epsilon_i = 1 \\ \overline{A_i} & \epsilon_i = -1. \end{cases}$$

DO 15.19. If $A_1, \ldots, A_5$ are independent events, then the events $A_1 \cup A_2, A_3 \setminus A_4, \overline{A_5}$ are also independent.

DO 15.20. If $X_1, \ldots, X_5$ are independent random variable, then any functions of disjoint blocks of these variables are also independent. For instance, $X_1 + \mathrm{e}^{X_2}, X_3^2 \cdot \sqrt{2^{X_4} + X_5^2}$ are independent random variables.

Theorem 15.21. *Let $X_1, \ldots, X_n$ be independent random variables. Then*

$$E\left(\prod_{i=1}^{n} X_i\right) = \prod_{i=1}^{n} E(X_i)\,.$$

Definition 15.22. A **Bernoulli trial** with probability p of success is a random variable X taking values 0 and 1 only, with

$$\begin{aligned} P(X=1) &= p \\ P(X=0) &= 1-p \end{aligned}$$

Notation 15.23. The abbreviation "i.i.d." stands for the phrase "independent, identically distributed."

Let $X_1, \ldots, X_n$ be i.i.d. Bernoulli random variables with probability p of success. Let $Y = \sum_{i=1}^{n} X_i$ denote the number of successes. We say that Y has **binomial** distribution with parameters n and p. The distribution of Y can be expressed as

$$P(Y=k) = \binom{n}{k} p^k (1-p)^{n-k}. \tag{2}$$

UNIMODAL AND LOG-CONCAVE SEQUENCES

Definition 15.24. A sequence $a_0, \ldots, a_n$ of real numbers is **unimodal** if there is a k such that $a_{i-1} \le a_i$ for all $i < k$ and $a_{i-1} \ge a_i$ for $i > k$.

Remark 15.25. A monotone non-increasing sequence is unimodal (take $k = 0$);
a monotone non-decreasing sequence is unimodal (take $k = n$).

Definition 15.26. A sequence $a_0, \ldots, a_n$ of real numbers is **strictly unimodal** if there is a k such that $a_{i-1} < a_i$ for all $i < k$ and $a_{i-1} > a_i$ for $i > k$.

Remark 15.27. Note that this definition permits $a_{k-1} = a_k$; these two are then the largest elements of the sequence.

HW+ 15.28. (9 points) Consider the binomial distribution (Eq. (2)). Prove that the sequence
$P(Y = k)$ $(k = 0, \ldots, n)$ is strictly unimodal. Show that the peak occurs at some a_k such that $|k - np| < 1$. At most two top values are equal.

We shall see shortly that the following stronger result holds.

Theorem 15.29. *If $X_1, \dots, X_n$ are independent (not necessarily i.i.d.) Bernoulli trials and $Y = \sum_{i=1}^n X_i$, then*

$$P(Y=0), P(Y=1), \dots, P(Y=n)$$

is unimodal.

Definition 15.30. A sequence $a_0, \dots, a_n$ of positive real numbers is **log-concave** if $(\forall i)(a_{i-1} \cdot a_{i+1} \le a_i^2)$.

HW 15.31. (5 points) If a sequence of positive reals is log-concave, then it is unimodal.

REAL-ROOTED POLYNOMIALS AND LOG-CONCAVITY

Definition 15.32. Let $f = a_0 + a_1 t + \cdots + a_n t^n$ be a polynomial of degree n with real coefficients. We say that f is **real-rooted** if all roots of f are real, i. e., f can be written as $f(t) = a_n \prod_{i=1}^n (t - \alpha_i)$ where $\alpha_i \in \mathbb{R}$.

Theorem 15.33. *Let $f(t) = a_0 + a_1 t + \cdots + a_n t^n$ be a real-rooted polynomial with non-negative coefficients Then the sequence $a_0, \dots, a_n$ is log-concave.*

Examples 15.34. The polynomial $(1+x)^n = \sum_{k=0}^n \binom{n}{k} x^k$ has all real roots (all are negative one), so by Theorem 15.33, the sequence $\binom{n}{k}$, $0 \le k \le n$, is log-concave. Similarly, the polynomial

$$\big((1-p)+px\big)^n = \sum_{k=0}^n \binom{n}{k} p^k (1-p)^{n-k} x^k$$

has all real roots ($x = -\frac{1-p}{p}$), so the values $\binom{n}{k} p^k (1-p)^{n-k}$, $0 \le k \le n$, are log-concave.

Theorem 15.35 (Newton's Inequalities)**.** *Let $f(x) = a_0 + a_1 x + \cdots + a_n x^n$ be a real-rooted polynomial with non-negative coefficients, satisfying $a_0 a_n \neq 0$. Then the sequence*

$$\frac{a_k}{\binom{n}{k}}, \quad 0 \le k \le n$$

is log-concave.

DO 15.36. If a sequence $\{a_k\}_{k=0}^n$ of positive numbers satisfies Newton's inqualities then the sequence is strictly log-concave. the sequence of coefficients a_k, $0 \le k \le n$, is strictly log-concave:

$$a_k^2 > a_{k-1} a_{k+1}.$$

Definition 15.37. Suppose $f(x) = a_0 + a_1 x + \cdots + a_n x^n$ and $a_0 a_n \neq 0$. Then $f^*(x) := a_n + a_{n-1} x + \cdots + a_0 x^n$ is the **reciprocal polynomial** to $f(x)$. In particular, $f^*(x) = x^n \cdot f(\frac{1}{x})$.

DO 15.38. Let $f(x) = a_0 + a_1 x + \cdots + a_n x^n$ be a real-rooted polynomial with non-negative coefficients and $a_0 a_n \neq 0$ with roots $(\alpha_i)_{i=1}^n$. Let f^* be the polynomial reciprocal to f. Then g is also real-rooted, having roots $(\frac{1}{\alpha_i})_{i=1}^n$.

HW+ 15.39. (6 points) Prove Newton's Inequalities. Hint. If $f(t)$ has degree n, let $g = \frac{d^{k-1}}{dt^{k-1}} f(t)$, and let h be the reciprocal to g. Let $\ell = \frac{d^{n-k-1}}{dt^{n-k-1}} h$. Then $\deg \ell = 2$. ℓ being real-rooted implies its discriminant is greater or equal to zero.

CHEBYSHEV POLYNOMIALS

The function $\cos(n\theta)$ is a polynomial of $\cos(\theta)$.

Definition 15.40. The polynomial T_n, called the **Chebyshev polynomial of the first kind**, is defined by the identity

$$\cos(n\theta) = T_n(\cos\theta)\,. \tag{3}$$

For example, the identity $\cos(2\theta) = 2(\cos\theta)^2 - 1$ yields the Chebyshev polynomial $T_2(t) = 2t^2 - 1$.

The function $\dfrac{\sin((n+1)\theta)}{\sin(\theta)}$ is also a polynomial of $\cos\theta$.

Definition 15.41. The polynomial U_n, called the **Chebyshev polynomial of the second kind**, is defined by the identity

$$\frac{\sin((n+1)\theta)}{\sin(\theta)} = U_n(\cos\theta)\,. \tag{4}$$

For example, the identity $\dfrac{\sin(2\theta)}{\sin(\theta)} = 2\cos\theta$ yields the Chebyshev polynomial $U_1(t) = 2t$.

Here are the first few Chebyshev polynomials.

$$\begin{array}{ll}
T_0(t) = 1 & U_0(t) = 1 \\
T_1(t) = t & U_1(t) = 2t \\
T_2(t) = 2t^2 - 1 & U_2(t) = 4t^2 - 1 \\
T_3(t) = 4t^3 - 3t & U_3(t) = 8t^3 - 4t \\
T_4(t) = 8t^4 - 8t^2 + 1 & U_4(t) = 16t^4 - 12t^2 + 1 \\
T_5(t) = 16t^5 - 20t^3 + 5t & U_5(t) = 32t^5 - 32t^3 + 6t
\end{array}$$

Theorem 15.42. *The Chebyshev polynomials satisfy the following two identical recurrences:*

$$T_{n+1}(t) = 2t \cdot T_n(t) - T_{n-1}(t) \qquad U_{n+1}(t) = 2t \cdot U_n(t) - U_{n-1}(t)$$

HW 15.43. (5 points) Prove these recurrences. Hint: use the following trigonometric identities:

$$\cos(\alpha+\beta) + \cos(\alpha-\beta) = 2\cos\alpha\cos\beta \tag{5}$$

(for T_n) and

$$\sin(\alpha+\beta) + \sin(\alpha-\beta) = 2\sin\alpha\cos\beta \tag{6}$$

(for U_n).

HW+ 15.44. (8 points) Prove:

$$f_{P_n}(t) = U_n(t/2)\,. \tag{7}$$

(Here f_{P_n} is the characteristic polynomial of the adjacency matrix of the path of length $n-1$.)

HW 15.45. (3+3 points) Find the roots of T_n and U_n. Your answer should be simple trigonometric formulas.

MATCHINGS POLYNOMIAL

Notation 15.46. For a graph G, let $m_k(G)$ denote the number of k-matchings, i. e., matchings of G consisting of k disjoint edges.

So $m_0(G) = 1$ and $m_1 = m$. The largest k for which $m_k \neq 0$ is $k = \nu(G)$, the matching number. For even n, the number $m_{n/2}$ counts perfect matchings.

Notation 15.47 (Matching generator function)**.**

$$m_G(t) = \sum_{i=0}^{\nu(G)} m_i(G) t^i\,.$$

The central result of the theory of this function was discovered in 1972 by statistical physicists Ole Heilmann and Elliot H. Lieb and published in their paper titled "The theory of monomer-dimer systems."

Theorem 15.48 (Heilmann and Lieb, 1972)**.** *The matching generating function is real-rooted.*

Corollary 15.49. *The sequence $m_0(G), m_1(G), \ldots, m_\nu(G)$ is strictly log-concave.*

The consequences of the reality of the roots go way beyond this log-concavity and are at the heart of Heilmann and Lieb's proof of the absence of phase transitions in the physical systems named in the title. of their paper.

For reasons that will soon become evident, a variant of this generating function is the preferred object of study.

Definition 15.50. The **matchings polynomial** of a graph G is the polynomial

$$\mu_G(t) = \sum_{k=0}^{\nu(G)} (-1)^k m_k t^{n-2k} = t^n - m_1 t^{n-2} + m_2 t^{n-4} - \ldots.$$

The connection between the matchings polynomial and the matching generator function is simple.

DO 15.51. $\mu_G(t) = t^{n-\nu} m_G^*(-t^2)$, where m_G^* is the reciprocal polynomial of m_G.

Theorem 15.52. *μ_G is real-rooted.*

DO 15.53. μ_G real-rooted $\iff$ m_G real-rooted.

Theorem 15.54. *If G is a tree then $\mu_G = f_G$, where f_G is the characteristic polynomial of the adjacency matrix.*

As we shall see, the matchings polynomials are related to several families of classical orthogonal polynomials:

$$\mu_{P_n}(t) = U_n(t/2)$$

$$\mu_{C_n}(t) = 2 \cdot T_n(t/2)$$

$$\mu_{K_n}(t) = He_n(t)$$

where He_n is the degree-n *Hermite polynomial*, to be defined later.

Orthogonal polynomials have been known to be real-rooted for a century and a half; this includes Chebyshev's polynomials and the Hermite polynomials. Moreover, each of these polynomial sequences shows interlacing of their roots. The same holds for the characteristic polynomials of the adjacency matrices of graphs and therefore for the matchings polynomials of trees. Theorem 15.48 is a remarkable generalization of these facts. Time permitting, we shall prove most statements made.