Graph Theory: CMSC 27530/37530 Lecture 16

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HW+ and **Bonus+** indicate homework and Bonus problems due a week from the date of the class in which most problems of the given problem set were assigned; in this case, due next Thursday.

SPECTRAL GRAPH THEORY AND CHEBYSHEV POLYNOMIALS

Notation 16.1. For a graph G, we refer to the eigenvalues of the adjacency matrix of G as the **eigenvalues of the graph** G and denote them by $\lambda_1(G) \geq \cdots \geq \lambda_n(G)$.

HW 16.2. (4 points) Let $H \subseteq G$ be a subgraph. Prove: $\lambda_1(G) \geq \lambda_1(H)$.

BONUS 16.3. (7 points) Let $H \subset G$ be a proper subgraph, i. e., $H \subseteq G$ and $H \neq G$. Assume G is connected. Prove: $\lambda_1(G) > \lambda_1(H)$.

BONUS 16.4. (5 points) Show that for $H \subseteq G$ it is possible that $\lambda_2(G)$ is much smaller than $\lambda_2(H)$. Specifically, for all sufficienty large n, find a graph–subgraph pair $H \subseteq G$ such that $\lambda_2(G) < 0$ while $\lambda_2(H) > n/3$, where n is the number of vertices of G.

HW 16.5. (4 points) If H is an *induced* subgraph of G, then

$$(\forall i)(\lambda_i(G) \geq \lambda_i(H)). \tag{1}$$

HW 16.6. (3+3 points) As usual, let $\alpha(G)$ denote the independence number and $\omega(G)$ the clique number of the graph G. Prove:

(a) $\lambda_{\alpha(G)} \geq 0$.

(b) $\lambda_{\omega(G)} \geq -1$.

Proposition 16.7. *The roots of U_n are* $\cos\left(\frac{k\pi}{n+1}\right)$, $1 \leq k \leq n$, *where U_n is the n-th Chebyshev polynomial of the second kind..*

Proof. Using the recurrence for U_n it follows by induction that $\deg(U_n) = n$. So we just need to verify that the n distinct numbers listed are indeed roots.

By definition, we have

$$U_n(\cos\theta) = \frac{\sin((n+1)\theta)}{\sin\theta}. \tag{2}$$

Trigonometry tells us that $\sin\alpha = 0 \iff \alpha = k\pi$ for an integer k. Therefore, the numerator $\sin((n+1)\theta) = 0 \iff \theta = \theta_k := \frac{k\pi}{n+1}$. Moreover, for $1 \le k \le n$, the denominator $\sin(\theta_k) \neq 0$ and therefore the quotient is zero, demonstrating that $U_n(\cos\theta_k) = 0$ for $k = 1, \ldots, n$. $\square$

DO 16.8. $\sin x \sim x$ as $x \to 0$, i. e., $\displaystyle\lim_{x\to 0} \frac{\sin x}{x} = 1$.

DO 16.9. $U_n(1) = n + 1$.

Proof. Since $\cos(0) = 1$, and $\sin((n+1)\cdot 0)/\sin 0$ is undefined, we need to rely on the continuity of the polynomial U_n. So instead of substituting $\theta = 0$, we let $\theta \to 0$. By the previous exercise, $U_n(1) = \displaystyle\lim_{t\to 1-} U_n(t) = \lim_{\theta\to 0} \frac{\sin((n+1)\theta)}{\sin\theta} \sim \frac{(n+1)\theta}{\theta} = n+1$ as $\theta \to 0$. $\square$

DO 16.10. The eigenvalues of P_n are the numbers $\quad 2\cos\left(\dfrac{k\pi}{n+1}\right), \quad 1 \le k \le n.$

Hint. According to one of the HW problems due Tuesday, we have $f_{P_n}(t) = U_n(t/2)$.

By an *all-positive* vector $\mathbf{x} = (x_1, \ldots, x_n)^T \in \mathbb{R}^n$ we mean a vector such that all coordinates x_i are positive.

HW 16.11. (6 points) Find a graph with $n = 100$ vertices that does not have an all-positive eigenvector. Prove.

HW+ 16.12. (6 points) Prove that $U_n(3/2)$ is a Fibonacci number. Which one? Recall that the Fibonacci numbers begin with $F_0 = 0$ and $F_1 = 1$. Do not use results we did not prove.

BONUS+ 16.13. (8 points) Consider the path P_{k+1} with vertices $v_0, \ldots, v_k$ in this order. Let H be a connected graph with $n - k$ vertices, including v_k but none of the other v_i ($i \le k-1$). Let G_n be the graph H with P_{k+1} attached at v_k as a "tail," so $V(G) = V(H) \cup \{v_0, \ldots, v_{k-1}\}$ and $E(G) = E(H) \cup E(P_{k+1})$. So G has n vertices and we have $\deg_G(v_0) = 1$, $\deg_G(v_i) = 2$ for $i = 1, \ldots, k-1$, $\deg_G(v_k) = 1 + \deg_H(v_k)$, and for $1 \le i \le k-1$, the two neighbors of v_i in G are v_{i-1} and v_{i+1}.

Let λ denote the largest eigenvalue of G_n and let $\mathbf{x} = (x_0, \ldots, x_{n-1})^T$ be an all-positive eigenvector of G_n such that for $0 \le i \le k$ the coordinate x_i corresponds to vertex v_i. Prove: $x_k/x_0 = U_k(\lambda/2)$.

BONUS+ 16.14. (8 points) Let G be a connected graph, λ_G its largest eigenvalue, and $\mathbf{x}_G = (x_1(G), \ldots, x_n(G))^T$ the (unique) all-positive eigenvector with unit norm. Let $x_{\min}(G) = \min_i x_i(G)$. We know that $\mathbf{x}_{\min}(G) > 0$, but how small can it be? Prove: $x_{\min}(G) \ge n^{-n}$.

CH 16.15. (9 points) How tight is the lower bound in the preceding exercise? Let $\xi_n = \min_G x_{\min}(G)$ where the minimum is taken over all connected graphs G with n vertices. The preceding exercise says that $\xi_n \geq n^{-n}$. Prove that this lower bound is not too bad: there exists a constant $c > 0$ such that for all sufficiently large n we have $\xi_n < n^{-cn}$. (The Greek letter ξ spells "xi" as in "oxigen" (LaTeX: `\xi`).)

INNER PRODUCTS

The **standard dot product** in $\mathbb{R}^n$ is a function $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$: it takes as input a pair $(\mathbf{a}, \mathbf{b})$ of vectors and assigns to them the value $\mathbf{a}^T\mathbf{b}$.

The dot product is an instance of the general concept of **inner products.**

Informally, a **vector space** over $\mathbb{R}$ is a set of objects we call "vectors," where linear combinations of vectors with real coefficients are defined and the usual rules hold. Please look up the formal definition. The examples that most matter for us are $\mathbb{R}^n$ and spaces of real functions such as $C[I]$, the set of continuous functions over the interval $I \subseteq \mathbb{R}$, and subspaces of these. Foremost among the latter is the space $\mathbb{R}[t]$ of polynomials with real coefficients (where t is the variable).

Terminology 16.16. In certain contexts, a function $F : D \to W$ is called a *form* if the codomain is $W = \mathbb{R}$. This is usually the case when D is represented as a direct product of vector spaces and F is described by homogeneous polynomials.

A function $F : V \times V \to \mathbb{R}$ is *linear in the first argument* if the following conditions hold.

(i) $F(\lambda\mathbf{a}, \mathbf{b}) = \lambda F(\mathbf{a}, \mathbf{b})$.

(ii) $F(\mathbf{a}_1 + \mathbf{a}_2, \mathbf{b}) = F(\mathbf{a}_1, \mathbf{b}) + F(\mathbf{a}_2, \mathbf{b})$.

Similarly, F is *linear in the second argument* if the following hold.

(iii) $F(\mathbf{a}, \lambda\mathbf{b}) = \lambda F(\mathbf{a}, \mathbf{b})$.

(iv) $F(\mathbf{a}, \mathbf{b}_1 + \mathbf{b}_2) = F(\mathbf{a}, \mathbf{b}_1) + F(\mathbf{a}, \mathbf{b}_2)$.

Definition 16.17. A function $F : V \times V \to \mathbb{R}$ is a **bilinear form** if it is linear in the first argument and linear in the second argument.

Example 16.18. Let $B \in M_n(\mathbb{R})$. Then the function $F_B : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by

$$F_B(\mathbf{x}, \mathbf{y}) = \mathbf{x}^T B\mathbf{y} \tag{3}$$

is a bilinear form.

DO 16.19. Prove that Eq. (3) defines all bilinear forms with domain $\mathbb{R}^n \times \mathbb{R}^n$: if $F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a bilinear form then there exists $B \in M_n(\mathbb{R})$ such that $F = F_B$.

Example 16.20. $C[a, b]$ denotes the space of continuous real functions defined on the closed interval $[a, b]$ $(a < b)$. The function $F : C[a, b] \times C[a, b]$ defined by $F(f, g) = \int_a^b f(t)g(t)dt$ is a bilinear form. More generally, let $w \in C[a, b]$ and define the function $F_w : C[a, b] \times C[a, b]$ as follows:

$$F_w(f, g) = \int_a^b f(t)g(t)w(t)dt\,. \tag{4}$$

Then F_w is a bilinear form. We refer to w as the **weight function**.

Definition 16.21. A form $F : V \times V \to \mathbb{R}$ is **symmetric** if $(\forall \mathbf{u}, \mathbf{v} \in V)(F(\mathbf{u}, \mathbf{v}) = F(\mathbf{v}, \mathbf{u}))$.

DO 16.22. The bilinear form F_B defined by Eq. (3) is symmetric if and only if the matrix B is symmetric. — The bilinear form F_w defined by Eq. (4) is always symmetric.

Definition 16.23. A symmetric bilinear form $F : V \times V \to \mathbb{R}$ is **positive semidefinite** if $(\forall \mathbf{x})(F(\mathbf{x}, \mathbf{x}) \geq 0)$. If in addition $F(\mathbf{x}, \mathbf{x}) > 0$ for all $\mathbf{x} \neq \mathbf{0}$, then F is **positive definite**.

Definition 16.24. An **inner product** on a vector space V is a positive definite bilinear form $V \times V \to \mathbb{R}$. We often use the notation $\langle \mathbf{a}, \mathbf{b} \rangle$ to denote the value of the inner product of vectors $\mathbf{a}, \mathbf{b} \in V$.

Definition 16.25. A symmetric matrix $B \in M_n(\mathbb{R})$ is **positive definite** if $(\forall \mathbf{x} \in \mathbb{R}^n)(\mathbf{x} \neq \mathbf{0} \implies \mathbf{x}^T B \mathbf{x} > 0)$. When we say "the matrix B is positive definite," we always assume that B is symmetric.

DO 16.26. The symmetric matrix $B \in M_n(\mathbb{R})$ is positive definite if and only if all of its eigenvalues are positive.

DO 16.27. Let $B \in M_n(\mathbb{R})$ be a symmetric matrix. Consider the symmetric bilinear form F_B on $\mathbb{R}^n$ defined by Eq. (3): $F_B(\mathbf{x}, \mathbf{y}) = \mathbf{x}^T B \mathbf{y}$ (cf. exercise DO 16.22). Prove: F_B is positive definite if and only if the matrix B is positive definite.

DO 16.28. Consider the symmetric bilinear form F_w on $C[a, b]$ defined by Eq. (4). Prove: F_w is positive definite if and only if the weight function w is non-negative and not everywhere zero in $[a, b]$.

In the rest of this section we assume that a positive definite symmetric bilinear form has been designated as the inner product; the inner product of vectors $\mathbf{a}, \mathbf{b} \in V$ is denoted $\langle \mathbf{a}, \mathbf{b} \rangle$.

Definition 16.29. Two vectors $\mathbf{a}, \mathbf{b}$ are **orthogonal** if $\langle \mathbf{a}, \mathbf{b} \rangle = 0$. We write $\mathbf{a} \perp \mathbf{b}$.

Definition 16.30. We define a **norm** on V by $\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$. We say that $\|\cdot\|$ is the **norm induced by the given inner product**.

DO 16.31 (Cauchy–Schwarz)**.** For all $\mathbf{a}, \mathbf{b} \in V$ we have

$$|\langle \mathbf{a}, \mathbf{b} \rangle| \leq \|\mathbf{a}\| \cdot \|\mathbf{b}\| .$$

Hint. Consider the function $f(t) = \|\mathbf{a} + t \cdot \mathbf{b}\|^2$ $(t \in \mathbb{R})$. Then $f(t) \geq 0$ for all t. Notice that $f(t)$ is a quadratic polynomial of t. Compute its discriminant.

DO 16.32 (Triangle inequality)**.** For all $\mathbf{a}, \mathbf{b} \in V$ we have

$$\|\mathbf{a} + \mathbf{b}\| \leq \|\mathbf{a}\| + \|\mathbf{b}\| .$$

Hint. Prove that this is equivalent to Cauchy–Schwarz.

HW 16.33. (5 points) Consider the space $C[0,2\pi]$ with the inner product

$$\langle f,g\rangle = \int_0^{2\pi} f(t)g(t)dt\,.$$

(So the weight function is the constant function $w(t)=1$.) Prove: the functions $1, \cos\theta, \cos(2\theta), \cos(3\theta), \dots$ are pairwise orthogonal. Find their norms.
Hint: $\cos(\alpha+\beta)+\cos(\alpha-\beta) = 2\cos\alpha\cdot\cos\beta$.

ORTHOGONAL POLYNOMIALS

Recall that $\mathbb{R}[t]$ denotes the set of polynomials with real coefficients.

Definition 16.34. The **degree** of the polynomial $f(t)=\sum_{i=0}^n a_i t^i$ is the largest value k such that $a_k \neq 0$. If all coefficients are zero (the zero polynomial) then the degree is $-\infty$.

DO 16.35. With this definition, the following two rules hold without exception.

(1) $\deg(f+g) \le \max\{\deg(f), \deg(g)\}$

(2) $\deg(fg) = \deg(f) + \deg(g)$

Notation 16.36. We write $\mathbb{R}^{\le j}[t]$ to denote the set of polynomials of degree at most j. Note that this set includes the zero polynomial.

$\mathbb{R}[t]$ is a vector space.

DO 16.37. $\mathbb{R}^{\le j}[t]$ is a subspace of $\mathbb{R}[t]$.

What is a basis of $\mathbb{R}[t]$? We want a list of polynomials such that each member of $\mathbb{R}[t]$ can be *uniquely* expressed as a linear combination of the polynomials on the list.

DO 16.38. Let $f_0, f_1, f_2, \dots \in \mathbb{R}[t]$ be a sequence of polynomials such that $\deg(f_i) = i$. Then this sequence forms a basis of $\mathbb{R}[t]$.

One simple choice is the basis $\mathcal{B} = \{1, t, t^2, t^3, \dots\}$. We shall refer to this sequence as the **standard basis** of $\mathbb{R}[t]$.

Consider an interval $[a,b]$ and a weight function $w \in C[a,b]$. Assume $(\forall t\in[a,b])(w(t)\ge 0)$ and $(\exists t\in[a,b])(w(t)>0)$. Then, according to exercise DO 16.28, Eq. (4) defines an inner product on $\mathbb{R}[t]$.

We relax the condition on w in two directions. First, we only require w to be defined (and continuous) in the interior of the interval: $w\in C(a,b)$. For this to work we need to add the condition

$$\int_a^b w(t)dt < \infty\,. \tag{5}$$

DO 16.39. If the non-negative, not identically zero weight function $w\in C(a,b)$ satisfies Eq. (5) then Eq. (4) defines a positive definite bilinear form on $\mathbb{R}[t]$.

The other direction in which we relax the condition is that we permit all of $\mathbb{R}$ to be the domain of w, so $w \in C(-\infty, \infty)$. In this case we need to make the following additional restriction on w: for every n, the $(2n)$-th moment of w is finite:

$$\int_{-\infty}^{\infty} t^{2n} w(t)dt < \infty\,. \tag{6}$$

More generally, we also permit an infinite interval of the form $I = (a, \infty)$ or $I = (-\infty, b)$, and require the integral over I to be finite.

DO 16.40. Let I be any (finite or infinite) open interval. If the non-negative, not identically zero weight function $w \in C(I)$ satisfies

$$\int_I t^{2n} w(t)dt < \infty\,. \tag{7}$$

then the equation

$$F_w(f, g) = \int_I f(t)g(t)w(t)dt\,. \tag{8}$$

defines a positive definite symmetric bilinear form on $\mathbb{R}[t]$.

For the rest of this section we shall fix a finite or infinite open interval $I \subseteq \mathbb{R}$ and a non-negative, not identically zero weight function $w \in C(I)$ that satisfies Eq. (7). Such a weight function defines a positive definite inner product on $\mathbb{R}[t]$ which we call the **inner product defined by** w.

Having thus fixed an inner product on $\mathbb{R}[t]$, we are interested in an orthonormal basis.

Definition 16.41. Let $f_0, f_1, \ldots$ be an infinite sequence of polynomials. We say that this is a **sequence of orthogonal polynomials** (with respect to I and w) if

(a) $(\forall k)(\deg f_k = k)$

(b) $(\forall i, j)(i \neq j \implies f_i \perp f_j)$

Multiplication of each f_i by a non-zero scalar retains the property of being a sequence of orthogal polynomials. Therefore WLOG we may additionally assume that all the f_i are monic.

DO 16.42. With this additional assumption, the sequence f_i is uniquely determined by the weight function w. Moreover, this unique sequence of orthogonal polynomials can be obtained by applying Gram–Schmidt orthogonalization to the standard basis of $\mathbb{R}[t]$.

Terminology 16.43. An **orthogonal polynomial** (with respect to a given interval I and weight function w) is a (not necessarily monic) polynomial that belongs to a sequence of orthogonal polynomials defined by I and w.

Classical examples of orthogonal polynomials follow in the next section.

DO 16.44. Prove that the weight function $w(t) = \dfrac{1}{\sqrt{1-t^2}}$ satisfies Eq. (5), and therefore also Eq. (7), on the interval $I = (-1, 1)$.

HW 16.45. (6 points) Prove: the weight function $w(t) = \dfrac{1}{\sqrt{1-t^2}}$ on the interval $I = (-1, 1)$ defines the orthogonal polynomials T_n, the Chebyshev polynomials of the first kind. Hint: change variables in exercise HW 16.33. Don't forget to verify that this weight function is permitted.

DO 16.46. The weight function $w(t) = \sqrt{1-t^2}$ on the interval $I = (-1, 1)$ defines the orthogonal polynomials U_n, the Chebychev polynomials of the second kind.

Let us now consider the weight function $w(t) = \mathrm{e}^{-t^2/2}$ on the entire real line. First we need to verify that the moments exist.

DO 16.47. Prove that the weight function $w(t) = \mathrm{e}^{-t^2/2}$ satisfies Eq. (6).

Definition 16.48. The Hermite polynomials $He_0, He_1, \dots$ are the monic orthogonal polynomials defined by the weight function $w(t) = \mathrm{e}^{-t^2/2}$ on the interval $I = \mathbb{R}$.

Definition 16.49. Confusingly, another family of polynomials, denoted $H_0, H_1, \dots$, are also called "Hermite polynomials." They are orthogonal w.r.t. the weight function $w(t) = \mathrm{e}^{-t^2}$ on $\mathbb{R}$.

Remark 16.50. The Hermite polynomials He_n are popular among probabilists, while the H_n are popular among physicists. One relates the Guassian density to its n-th derivative:

$$He_n(t)\mathrm{e}^{-t^2/2} = (-1)^n \frac{d^n}{dt^n}\mathrm{e}^{-t^2/2}\,. \tag{9}$$

The other solves the Schrödinger equation for a quantum harmonic oscillator. The two forms are related by the equation

$$H_n(t) = 2^{n/2} He(2^{n/2}t). \tag{10}$$

When speaking of Hermite polynomials, in this class we shall always mean the probabilists' sequence He_n. These polynomials obey the following recurrence.

$$He_{n+1}(t) = t \cdot He_n(t) - nHe_{n-1}(t). \tag{11}$$

Here are the first few.

$$\begin{aligned}
He_0(t) &= 1\\
He_1(t) &= t\\
He_2(t) &= t^2 - 1\\
He_3(t) &= t^3 - 3t\\
He_4(t) &= t^4 - 6t^2 + 3\\
He_5(t) &= t^5 - 10t^3 + 15t
\end{aligned}$$

ORTHOGONAL POLYNOMIALS: REAL-ROOTED, INTERLACING

The following remarkable result holds for all sequences of orthogonal polynomials, regardless of the weight function.

Theorem 16.51. *Let $f_0, f_1, \ldots$ be a sequence of orthogonal polynomials (having fixed the interval I and the weight function w). Then*

(1) The f_i are real-rooted, and all roots are distinct.

(2) The roots of f_{k-1} strictly interlace the roots of f_k.

Strict interlacing means $\lambda_i > \mu_i > \lambda_{i+1}$ where the λ_i are the roots of f_k and the μ_i the roots of f_{k-1}.

The proof will be based on the following lemma.

Lemma 16.52 (3-term recurrence)**.** *If $f_0, f_1, \ldots$ are orthogonal polynomials, then*

$$(\forall n)(\exists \alpha_n, \beta_n, \gamma_n \in \mathbb{R})(f_{n+1}(t) = (\alpha_n t + \beta_n) f_n - \gamma_n f_{n-1})$$

where $\alpha_n \cdot \gamma_n > 0$.

BONUS 16.53. (10 points) Prove the lemma. You may assume without loss of generality that all the f_i are monic. Show that under this assumption $\alpha_n = 1$ and $\gamma_n > 0$.

Proof of Theorem 16.51 using the 3-term recurrence. We prove the two statements simultaneously by induction on n. Assume (1) and (2) hold up to degree n. Let $\lambda_1 > \lambda_2 > \ldots > \lambda_n$ be the roots of f_n and $\mu_1 > \mu_2 > \ldots > \mu_{n-1}$ the roots of f_{n-1}. We need to show (1) and (2) hold for f_{n+1}. Assume all the f_i are monic. In particular, $f_0 = 1$ and for $i \geq 2$ we have

$$\lim_{t\to\infty} f_i(t) = \infty\,. \tag{12}$$

Since $\mu_1 < \lambda_1$ and Eq. (12) holds for $i = n-1$, it follows from the intermediate value theorem that $f_{n-1}(\lambda_1) > 0$. But then, again by the intermediate value theorem, $f_{n-1}(t) < 0$ for $\mu_2 < t < \mu_1$. In particular, $f_{n-1}(\lambda_2) < 0$. Continuing in the same fashion, or better yet, by induction on i, we can prove that

$$\operatorname{sgn}(f_{n-1}(\lambda_i)) = (-1)^{i+1}\,.$$

By the 3-term recurrence with $\alpha_n = 1$ (because the f_i are monic), we have

$$f_{n+1}(\lambda_i) = (t + \beta_n) f_n(\lambda_i) - \gamma_n f_{n-1}(\lambda_i) = -\gamma_n f_{n-1}(\lambda_i)\,.$$

Since $\gamma_n > 0$, it follows that

$$\operatorname{sgn}(f_{n+1}(\lambda_i)) = -\operatorname{sgn}(f_{n-1}(\lambda_i)) = (-1)^i\,. \tag{13}$$

In particular, $f_{n+1}(\lambda_1) < 0$. Therefore, since Eq. (12) holds for $i = n+1$, the polynomial f_{n+1} must have a root that is greater than λ_1. Moreover, it must have a root strictly between λ_i and λ_{i-1} because it has opposite signs at the two endpoints of this interval. Finally, it must have a root that is less than λ_n because $\operatorname{sgn}(f_{n+1}(\lambda_n)) = (-1)^n$ but $\lim_{t\to-\infty} f_{n+1}(t) = (-1)^{n+1}\infty$. So we found $n+1$ distinct roots strictly interlaced with the roots of f_n; therefore this is the complete set of roots of f_{n+1}, completing the inductive step. (We used the IVT in every step.) $\square$

DO 16.54. Review the proof.

MATCHINGS POLYNOMIALS: REAL-ROOTED, INTERLACING

Remarkably, all matchings polynomials share the properties of orthogonal polynomials described in Theorem 16.51.

Theorem 16.55 (Heilmann and Lieb, 1972)**.** *Let μ_G denote the matchings polynomial of the graph G.*

(1) μ_G is real-rooted.

(2) For any vertex v, the roots of μ_G and μ_{G-v} interlace.

Let us recall the definition of the matchings polynomial μ_G. Let $m_k(G) = \#$ of k-matchings.

$$\mu_G(t) = \sum_{k=0}^{\nu(G)} (-1)^k m_k(G) t^{n-2k} \,. \tag{14}$$

For the proof of Theorem 16.55, we take the intuition from orthogonal polynomials. We start with an analogue of the 3-term recurrence.

Lemma 16.56 (3-term recurrence for the matchings polynomials)**.** *For any $v \in V$,*

$$\mu_G(t) = t \cdot \mu_{G-v}(t) - \sum_{w:w\sim v} \mu_{G-v-w}(t) \,.$$

HW 16.57. (9 points) Prove Lemma 16.56. Start with writing a recurrence for the numbers m_k: count the matchings that do not cover v, and those that do. The latter contain exactly one edge of the form $\{v, w\}$. Now do an accurate accounting for which matching numbers are multiplied by what powers of t and what sign they get.

DO 16.58. Derive Theorem 16.55 from Lemma 16.56.

HW 16.59. (7 points) If T is a tree, then $f_T = \mu_T$. (Here f_T is the characteristic polynomial of the adjacency matrix of T.)
Hint. Pick a vertex v of degree 1. Use Lemma 16.56 to obtain a recurrence for the matchings polynomial. Prove the same recurrence for the characteristic polynomial.

HW 16.60. (5 points) Prove that $\mu_{K_n} = He_n$.
Hint. Verify that the recurrence for the Hermite polynomials, Eq. (11), holds for μ_{K_n}.

BONUS+ 16.61. (9 points) Prove: $\mu_{C_n}(t) = 2 \cdot T_n(t/2)\,.$

DO 16.62. Based on the preceding exercise, show that the roots of μ_{C_n} are

$$\cos\left(\frac{(2k+1)\pi}{2n}\right) \quad \text{for} \quad k = 0, 1, \ldots, n-1\,.$$

INDEPENDENCE OF EVENTS VS. THE SIZE OF THE SAMPLE SPACE

For challenge problems, **don't look them up.** If you did, please acknowledge and you get partial credit. — Each of the problems below has an elegant solution – short, clean, convincing. Elegance counts.

Definition 16.63. Let (Ω, P) be a finite probability space. An event $A \subseteq \Omega$ is **trivial** if $P(A) = 0$ or 1.

DO 16.64. If $A_1, \ldots, A_k$ are independent events and A_{k+1} is a trivial event then $A_1, \ldots, A_{k+1}$ are independent.

DO 16.65. If there exist k nontrivial independent events in (Ω, P) then $|\Omega| \geq 2^k$.

CH 16.66. (6+7 points)

(a) For every $k \geq 3$, construct a probability space of size $|\Omega| = k + 1$ that contains k pairwise independent nontrivial events.

(b) For every k, construct a probability space of size $|\Omega| = O(k)$ that contains k pairwise independent events of probability 1/2 each.

CH 16.67. (10 points) Prove: If there exist k pairwise independent nontrivial events in (Ω, P) then $|\Omega| \geq k + 1$.

CH 16.68. (8 points) For every k, construct a probability space of size $|\Omega| = O(k)$ that contains k triplewise independent nontrivial events.

Triplewise independence means every 3 of them are independent. k-wise independence is defined analogously.

CH 16.69. (13 points) Prove: If in (Ω, P) there exist k nontrivial events that are four-wise independent then $|\Omega| \geq \binom{k+1}{2}$.