Graph Theory: CMSC 27530/37530 Lecture 17

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Revised by instructor

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HW+ and **Bonus+** indicate homework and Bonus problems due a week from the date of the class in which most problems of the given problem set were assigned; in this case, due next Tuesday.

Do not forget: the following previously assigned problems are also due Thursday, May 30: HW 16.12 and Bonus problems 16.13, 16.14, 16.61.

INDEPENDENT RANDOM VARIABLES, COVARIANCE, VARIANCE
CHEBYSHEV'S AND MARKOV'S INEQUALITIES

Let (Ω, P) be a probability space. Random variables $X_1, \dots, X_k : \Omega \to \mathbb{R}$ are independent if

$$(\forall \alpha_1, \dots, \alpha_k \in \mathbb{R}) \left(P\left(\bigwedge_{i=1}^{k} X_i = \alpha_i \right) = \prod_{i=1}^{n} P(X_i = \alpha_i) \right) . \tag{1}$$

Random variables satisfying Eq. (1) are sometimes called *fully independent* or *mutually independent* to emphasize the distinction of independence from pairwise independence.

The most important aggregate of a random variable X is the *expected value*, defined

$$E(X) = \sum_{a \in \Omega} X(a) \cdot P(a) . \tag{2}$$

DO 17.1. If X, Y are independent then $E(X \cdot Y) = E(X) \cdot E(Y)$.

Definition 17.2. The **covariance** of random variables X, Y is defined

$$\mathrm{Cov}(X, Y) = E(X \cdot Y) - E(X) \cdot E(Y) . \tag{3}$$

DO 17.3. If X, Y are independent then $\mathrm{Cov}(X, Y) = 0$.

Definition 17.4. Random variables X, Y are **uncorrelated** if $\mathrm{Cov}(X, Y) = 0$.

So if X, Y are independent then they are uncorrelated. The converse is false.

HW 17.5. (6 points) Find random variables X, Y which are uncorrelated but not independent. Make $|\Omega|$ as small as possible. — First you have to state your sample space and the probability distribution. Then you need to define your random variables X, Y, calculate $E(X), E(Y)$, and $E(XY)$, show that X, Y are uncorrelated. Finally you need to prove they are not independent.

DO 17.6. If $X_1, \ldots, X_k$ are independent then $E\big(\prod_i X_i\big) = \prod_i E(X_i)$.

How much does the random variable X tend to deviate from its expected value? One measure of this is the *mean deviation*: $E(|X - E(X)|)$. For reasons of mathematical simplicity, we find it much easier to work with a related quantity, the *variance*.

Definition 17.7. The **variance** of a random variable X is the quantity

$$\mathrm{Var}(X) = E\Big((X - E(X))^2\Big) .$$

DO 17.8. $\mathrm{Var}(X) \geq 0$. Prove: The variance is zero if and only X is almost constant, i. e., there is a number r such that $P(X = r) = 1$.

DO 17.9. $\mathrm{Var}(X) = E(X^2) - (E(X))^2$.

DO 17.10. Notice that $\mathrm{Var}(X) = \mathrm{Cov}(X, X)$.

Corollary 17.11 (Cauchy–Schwarz inequality)**.** $E(X^2) \geq (E(X))^2$.

DO 17.12. Show that the inequality in the preceding problem is equivalent to the following form of the Cauchy–Schwarz inequality. For vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^k$,

$$|\mathbf{x}^T\mathbf{y}| \leq \|\mathbf{x}\| \cdot \|\mathbf{y}\| \ . \tag{4}$$

HW 17.13. (4 points) $\mathrm{Var}\left(\sum_{i=1}^n X_i\right) = \sum_i \sum_j \mathrm{Cov}(X_i, X_j)$.

Corollary 17.14. *If $X_1, \ldots, X_k$ are pairwise independent, then*

$$\mathrm{Var}\left(\sum_i X_i\right) = \sum_i \mathrm{Var}(X_i) .$$

HW 17.15. (3+6 points) Let $G = ([n], E)$ be a 3-regular graph. Let $X_1, \ldots, X_n$ be independent unbiased Bernoulli trials (probability of success = 1/2). For every edge $e = \{i, j\} \in E$ let $Y_e = X_i X_j$. Let $Z = \sum_{e \in E} Y_e$. Determine

(a) $E(Z)$

(b) $\mathrm{Var}(Z)$.

You answers should be simple expressions in terms of n.

Proposition 17.16 (Markov's Inequality)**.** *Let X be a nonnegative random variable. Then for all $a > 0$ we have*

$$P(X \geq a) \leq \frac{E(X)}{a} . \tag{5}$$

Proof.

$$E(X) = \sum_{t \in \mathbb{R}^+} t \cdot P(X = t) \geq \sum_{t \geq a} t \cdot P(X = t)$$

$$\geq a \cdot \sum_{t \geq a} P(X = t) = a \cdot P(X \geq a).$$

The result now follows by substituting $a \cdot E(X)$ in place of a. ☐

Definition 17.17. The **standard deviation** of X is $\sigma(X) = \sqrt{\text{Var}(X)}$.

Proposition 17.18 (Chebychev's Inequality)**.** *For any random variable X and any $b > 0$ we have*

$$P\Big(|X - E(X)| \geq b\Big) \leq \frac{\text{Var}(X)}{b^2} = \left(\frac{\sigma(X)}{b}\right)^2 . \tag{6}$$

Proof. Let $Y = (X - E(X))^2$, so $Y \geq 0$. By Markov's inequality,

$$P(Y \geq b^2) \leq \frac{E(Y)}{b^2} = \frac{\text{Var}(X)}{b^2}.$$

☐

This is the simplest example of a **concentration inequality:** it says that that the value of X tends to be close to its expected value as long as the standard deviation is small.

PROBABILLITY GENERATING FUNCTIONS AND REAL-ROOTED POLYNOMIALS

Today we shall see further far-reaching consequences of real-rootedness.

Definition 17.19. The **generating function** of a sequence $\mathbf{a} = (a_0, a_1, \dots)$ is the function $f_{\mathbf{a}}(t) = \sum_k a_k t^k$. If the sequence is finite then $f_{\mathbf{a}}$ is a polynomial.

DO 17.20. Prove that the generating function of the Fibonacci numbers has the following closed-form expression:

$$\sum_{k=0}^{\infty} F_k t^k = \frac{t}{1 - t - t^2} . \tag{7}$$

Our main interest will be in the **matching generating function**

$$m_G(t) = \sum_{k=1}^{\nu(G)} m_k(G) \cdot t^k , \tag{8}$$

where $m_k(G)$ is the number of k-matchings of the graph G. By the Heilmann–Lieb theorem, this polynomial is real-rooted. We shall indicate far-reaching implications of this fact on the distribution of the numbers $m_k(G)$; under general conditions on the graph G, these will be shown to be asymptotically normal.

Definition 17.21. Let X be a random variable. If $\text{range}(X) \subseteq \mathbb{N} = \{0, 1, 2, \dots\}$, then we shall say that X is a **counting variable**.

Remark 17.22. "Counting variable" is not a standard term. Even though the concept it describes is the most frequent type of discrete random variable, I could not find a commonly used term for it. The caveat is that if you use this term outside this class, you need to define it.

Definition 17.23. The **probability generating function** of a counting variable X is defined by

$$f_X(t) = \sum_{k=0}^{\infty} P(X = k) \cdot t^k \,.$$

DO 17.24. If f_X is a probability generating function then $f_X(1) = 1$.

HW 17.25. (5 points) If X, Y are independent counting variables, then $f_{X+Y} = f_X \cdot f_Y$.

DO 17.26. More generally, if $X_1, \ldots, X_n$ are independent counting variables then

$$f_{\sum X_i} = \prod f_{X_i} \,.$$

The simplest examples of counting variables are Bernoulli trials; they take value 0 and 1 only.

DO 17.27. If X is a Bernoulli trial with probability p of success then

$$f_X = (1 - p) + pt \,. \tag{9}$$

Corollary 17.28. *Let X_i be independent Bernoulli random variables with probability of success p_i, and let $Y = \sum_{i=1}^{n} X_i$. Then the probability generating function of Y is*

$$f_Y = \prod_{i=1}^{n} \big((1 - p_i) + p_i t\big) \tag{10}$$

DO 17.29. This polynomial is real-rooted; the roots are the negative numbers $-\alpha_i$ where

$$\alpha_i = \frac{1 - p_i}{p_i} = \frac{1}{p_i} - 1 \,. \tag{11}$$

Let $g(t) = \prod_{i=1}^{n}(t + \alpha_i)$. Then

$$f_Y(t) = \frac{g(t)}{g(1)} \,. \tag{12}$$

We can read this observation backwards: instead of defining Y as a sum of independent Bernoulli trials, we can decompose a given counting variable Y into the sum of independent Bernoulli trials as long as f_Y is real-rooted.

Corollary 17.30. *If Y is a counting variable and f_Y is real-rooted, of degree n, then Y is a sum of n independent Bernoulli trials.*

More precisely, the distribution of Y is identical with the distribution of the sum of n independent Bernoulli trials. Note that these Bernoulli trials will usually not be identically distributed.

Proof. Since f_Y has non-negative coefficients, its roots muct be negative; let us denote them $-\alpha_i$ where $\alpha_i > 0$. Let $g(t) = \prod_{i=1}^{n}(t+\alpha_i)$. Then f_Y has the same roots (with the same multiplicities) as g and therefore g and f_Y differ only in a scalar factor: $f_Y(T) = c \cdot g(t)$. Given that $f_Y(1) = 1$, it follows that $c = 1/g(1)$. Therefore

$$f_Y(t) = \frac{g(t)}{g(1)} = \prod_{i=1}^{n} \frac{t+\alpha_i}{1+\alpha_i} = \prod_{i=1}^{n}(1-p_i+p_it) \tag{13}$$

where

$$p_i = \frac{1}{1+\alpha_i}\,. \tag{14}$$

(**DO:** Verify the last equation!) Note that $0 < p_i < 1$ (since $\alpha_i > 0$), so we can view p_i as the probability of success of a Bernoulli trial. But according to Cor. 17.28, the right-hand side of Eq. (13) is precisely the probability generating function of the sum of independent Bernoulli trials X_i with probability p_i of success. □

Definition 17.31. For a graph G, let M_G be the set of matchings. This will be our sample space, with uniform distribution. Let $X_G : M_G \to \mathbb{R}$ be the random variable that counts the edges of the matching $x \in M_G$. We call X_G the **matching counting variable** for G.

We now derive a powerful corollary of the Heilmann–Lieb theorem, the reality of the roots of the matching generating function.

Corollary 17.32. *The matching counting variable is a sum of independent Bernoulli trials.*

Proof. Let $m_G(t)$ be the matching generating function defined by Eq.(8), and let X_G be the matching counting variable. Now $|M_G| = \sum_k m_k(G) = m_G(1)$, so the probability generating function of X_G is

$$f_{X_G}(t) = \frac{m_G(t)}{m_G(1)}\,. \tag{15}$$

(**DO:** Verify!) So this polynomial is real-rooted, and the result follows from Cor. 17.30. □

CENTRAL LIMIT THEOREMS

Definition 17.33. Let Y be a random variable with $\sigma(Y) > 0$ (so Y is not almost constant). Then the **centered** version of Y is the variable $U = Y - E(Y)$ and the **normalized** version of Y is the variable

$$Z = \frac{Y - E(Y)}{\sigma(Y)}. \tag{16}$$

DO 17.34. (a) $E(U) = 0$ and $\sigma(U) = \sigma(Y)$

(b) $E(Z) = 0$ and $\sigma(Z) = 1$

(c) Show that among all random variables of the form $aY+b$ ($a,b\in\mathbb{R}$), the normalized version Z is the only one that satisfies (b).

HW+ 17.35. (6 points) Let X be a Bernoulli trial with success probability p ($0<p<1$) and let U be the centered version of X. Prove: $E(|U|^3) < \mathrm{Var}(X)$.

DO 17.36. Let X be a Bernoulli trial with success probability p ($0<p<1$) and let Z be the normalized version of X. Determine Z.

Solution. We have $E(X)=p$ and $\mathrm{Var}(X)=p(1-p)$, so

$$Z = \frac{X-p}{\sqrt{p(1-p)}} = \begin{cases} \sqrt{\frac{1-p}{p}} & \text{with probability } p \\ \sqrt{\frac{p}{1-p}} & \text{with probability } 1-p \end{cases} \tag{17}$$

□

Definition 17.37. The **cumulative distribution function** (CDF) of a random variable X is the function

$$F_X(t) = P(X \le t).$$

Definition 17.38. The **standard normal distribution** is the distribution defined by the CDF

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \mathrm{e}^{-t^2/2}\, dt\ . \tag{18}$$

Remark 17.39. The "density" of this distribution is the function $\Phi'(t) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-t^2/2}\, dt$, the standard bell curve.

Definition 17.40. Let $\{X_n\}$ be an infinite sequence of random variables, each defined on its own separate probability space. Assume the X_n are not almost constant; let Z_n denote the normalized version of X_n. Let F_n denote the CDF of Z_n. We say that the sequence $\{X_n\}$ is **asymptoticaly normal** if F_n approaches Φ uniformly, i. e., the distance $\sup_x |F(x)-\Phi(x)|$ approaches zero as $n\to\infty$.

Remark 17.41. The classical **Central Limit Theorem** (De Moivre–Laplace) says that for the binomial distributions are asymptotically normal in the following sense. Fix $0<p<1$ and let $X_1, X_2, \dots, X_n$ be independent Bernoulli trials with success probability p. Let $Y_n = \sum_{i=1}^n X_i$. Then the infinite sequence $\{Y_n\}$ is asymptotically normal.

The following result is one of many generalizations of the classical Central Limit Theorem.

Theorem 17.42 (Andrew Berry, Carl-Gustav Esséen, 1941/42)**.** *Let $(X_i)_{i=1}^n$ be independent random variables with respective standard deviation $\sigma(X_i)=\sigma_i$, and let ρ_i denote the third moment of $U_i = X_i - E(X_i)$, defined by $\rho_i = E(|U_i|^3)$. Let $Y=\sum_{i=1}^n X_i$ and let Z be the normalized version of Y. Let $\sigma := \sigma(Y) = \sqrt{\sum_{i=1}^n \sigma_i^2}$. Then*

$$(\forall x\in\mathbb{R})\left(|F_Z(x)-\Phi(x)| < \sigma^{-3}\cdot\sum_i \rho_i\right)\ . \tag{19}$$

The Berry–Esséen Theorem strengthens the classical result in several directions.

(a) It does not require the X_i to be Bernoulli trials

(b) It does not require the X_i to be identically distributed

(c) It gives a specific rate of convergence.

For our purposes, (a) will be irrelevant (our variables will be Bernoulli trials), but (b) and (c) are crucial.

Corollary 17.43. *Let $(X_i)_{i=1}^n$ be independent Bernoulli trials with respective success probability p_i. Let $Y = \sum_{i=1}^n X_i$ and let Z be the normalized version of Y. Let $\sigma := \sigma(Y) = \sqrt{\sum_{i=1}^n p_i(1-p_i)}$. Then*

$$(\forall x \in \mathbb{R}) \, (\, |F_Z(x) - \Phi(x)| < 1/\sigma \,) \, . \tag{20}$$

Proof. By exercise HW 17.35 we have $\rho_i < \sigma_i^2$. So the right-hand side of Eq. (19) is

$$\frac{\sum_i \rho_i}{\sigma^3} < \frac{\sum_i \sigma_i^2}{\sigma^3} = \frac{1}{\sigma} \, .$$

□

The following is an immediate corollary. (**DO:** Why?)

Corollary 17.44. *For $n = 1, 2, \ldots$ let Y_n be a sum of independent Bernoulli trials. If $\lim_{n\to\infty} \sigma_n = \infty$ then the sequence $Y_1, Y_2, \ldots$ is asymptotically normal.*

Remark 17.45. Note that the Bernoulli trials of which Y_n is the sum must be independent but they do not need to be identically distributed.

Remark 17.46. A conceptual clarification. The probability space for Y_n is the same as the probability space for the Bernoulli trials of which Y_n is the sum. However, the probability spaces associated with Y_n for distinct values of n are unrelated.

ASYMPTOTIC NORMALITY OF RANDOM MATCHINGS

Recall the matching generator function

$$m_G(t) = \sum_{i=0}^{\nu(G)} m_k t^k, \tag{21}$$

where m_k is the number of k-matchings.

The result that μ_G is real-rooted implies that m_G is also real-rooted. By Newton's inequalities, it follows that the sequence m_k, $k = 1, \ldots, \nu(G)$ is log-concave, and hence they are unimodal. We shall see that much more cn be said about the behavior of this sequence. Let us first consider a simple example.

Example 17.47. Let G be the graph consisting of $n/2$ disjoint edges: $G = \frac{n}{2} \cdot K_2$. Then $m_k = \binom{n/2}{k}$.

So the sequence of these sequences is asymptotically normal by De Moivre–Laplace. A far-reaching generalization of this fact was observed by Chris Godsil in 1981. For a graph G let X_G denote the matching counting variable defined in Def. 17.31, so X_G is the size of a random matching of G. Let $\sigma(G) := \sigma(X_G)$. Let us refer to the sequence $(m_0(G), m_1(G), \dots, m_{\nu(G)}(G))$ as the *matching sequence* of G.

Theorem 17.48 (Godsil, 1981)**.** *Let G_n be an infinite sequence of graphs. If $\sigma(G_n) \to \infty$ then the matching sequences of the G_n are asymptotically normal.*

DO 17.49. Show that this result follows by combining Cor. 17.32 (a consequence of the Heilmann–Lieb theorem) and Cor. 17.44 (a consequence of the Berry–Esséen theorem).

DO 17.50. Express $\sigma(G)$ in terms of the $m_k(G)$.

Godsil also gave rather general sufficient conditions that guarantee $\sigma(G_n) \to \infty$. One of these is the following.

Theorem 17.51 (Godsil)**.** *Let G_n be an infinite sequence of graphs without isolated vertices (i. e., $\deg_{\min} \geq 1$). If $\deg_{\max}(G_n)/|V(G_n)| \to 0$ then $\sigma(G_n) \to \infty$ and therefore the matching sequences of the G_n are asymptotically normal.*

Even though the conditions of this theorem do not hold for the complete graphs, Godsil showed that the complete graphs also satisfy the conclusion; in particular, the coefficients of the Hermite polynomials are asymptotically normal.
Recall that X_G is the *size of a random matching* (picked uniformly from M_G). Godsil comments that a necessary condition for $\sigma(G_n) \to \infty$ is $E(X_{G_n}) \to \infty$. Another condition that is obviously necessary is that $\nu(G_n) \to \infty$. He then comments that "interestingly enough, the second of these conditions implies the first" and cites an observation by this instructor that proves this.

Lemma 17.52 (Babai)**.** *The expected size of a random matching of G is at least $\nu(G)/3$, i. e., $E(X_G) \geq \nu(G)/3$.*

CH 17.53. Find a simple proof of this statement.

BONUS+ 17.54. (6 points) Find an infinite sequence $\{G_n\}$ of graphs such that $\nu(G_n) \to \infty$ but $\sigma(G_n) \to 0$.

Jeff Kahn (1998) significantly expanded Godsil's study of the asymptotic behavior of the matching sequence, including cases when $\sigma(G)$ is bounded and the matching sequence is asymptotically Poisson.

The presentation so far in this class was based on the following paper.

Chris D. Godsil, Matching behaviour is asymptotically normal. *Combinatorica* **1(4)** (1981) 369–376.

RAMSEY THEORY

Recall the Erdős–Szekeres theorem:

$$\binom{k+\ell}{k} \to (k+1, \ell+1)\,. \tag{22}$$

The diagnal case ($k = \ell$) gives

$$4^k > \binom{2k}{k} \to (k+1)_2.$$

Substituting $n = 4^k$, i. e., $k = \frac{1}{2}\log_2 n$, we obtain

$$n \to \left(1 + \frac{1}{2}\log_2 n\right)_2 . \tag{23}$$

The question is, how far the $(1/2)\log_2 n$ from the best possible bound. Paul Erdős showed in 1949 that the order of magnitude is correct.

Theorem 17.55 (Erdős). $n \not\to (1 + 2\log_2 n)_2$.

DO 17.56. Show that Theorem 17.55 is equivalent to the following statement.

- For all n there exists a graph G_n with n vertices such that

$$\max(\alpha(G_n), \alpha(\overline{G}_n)) < 1 + 2\log_2 n\,. \tag{24}$$

Erdős's proof was an early triumph of his "probabilistic method." Rather than constructing such graphs, he showed that **almost all graphs** have the required property. What this means is that the probability that a random graph on n vertices satisfies Eq. (24) approaches 1 as $n \to \infty$.

Theorem 17.57 (Erdős). *Let G_n be a graph on n vertices chosen uniformly at random. Then*

$$P\Big(\max(\alpha(G_n), \alpha(\overline{G}_n)) \geq 1 + 2\log_2 n\Big) \to 0 \text{ as } n \to \infty.$$

Proof. Fix a set V of n vertices. Our sample space Ω will be the set of all graphs with V as their set of vertices; the number of such graphs is $|\Omega| = 2^{\binom{n}{2}}$. Our probability distribution will be uniform over Ω. In other words, we pick our graph G "uniformly at random" from Ω.

Let $A \subseteq V$ be a set of size $|A| = k$. Then

$$P(A \text{ is independent}) = \frac{1}{2^{\binom{k}{2}}}.$$

Using the union bound,

$$P(\alpha(G) \geq k) \leq \sum_{A \subseteq V : |A| = k} P(A \text{ is independent}) = \binom{n}{k}\frac{1}{2^{\binom{k}{2}}}.$$

Combining this result with DO 17.59, we have

$$P(\alpha(G) \geq k) \leq \frac{1}{k!} \cdot \frac{n^k}{2^{k(k-1)/2}} = \frac{1}{k!}\left(\frac{n}{2^{(k-1)/2}}\right)^k.$$

If $k \geq 1 + 2\log_2 n$, then $2^{(k-1)/2} \geq n$, so we have

$$P(\alpha(G) \geq k) \leq \frac{1}{k!}.$$

The same statements apply to $\overline{G}$. Making another appeal to the union bound, we obtain that if $k \geq 1 + 2\log_2 n$ then

$$P(\alpha(G) \geq k \text{ or } \alpha(\overline{G}) \geq k) \leq \frac{2}{k!} \to 0 \text{ as } n \to \infty.$$

Thus, for almost all graphs ($P \to 1$), $\max(\alpha(G), \alpha(\overline{G})) < 1 + 2\log_2 n$. □

DO 17.58. Let us choose the graph G uniformly at random among the $2^{\binom{n}{2}}$ graphs on a given set of n vertices. For each pair $\{i, j\}$ of vertices ($i < j$) let $X(i, j)$ denote the Bernoulli trial that is successful if $\{i, j\} \in E(G)$. Prove that the probability of success is $1/2$ and that these $\binom{n}{2}$ Bernoulli trials are independent.

DO 17.59. $\displaystyle\binom{n}{k} \leq \frac{n^k}{k!}$.

BONUS+ 17.60. (6 points) Prove: almost all graphs have diameter 2. Define what this statement means.

BONUS+ 17.61. (7 points) Prove: if G does not contain K_5 then $\chi(G) = O(n^{3/4})$.
Hint. Erdős–Szekeres.

BONUS+ 17.62 (Explicit Ramsey graph by Zsigmond Nagy, 1973)**. (7 points)** Give a constructive proof of the relation $\displaystyle\binom{k}{3} \not\to (k+1)_2$ using the following graph. Let $V = \binom{[k]}{3}$ (the set of 3-subsets of $[k]$) be the set of vertices. Make $A, B \in V$ adjacent if $|A \cap B| = 1$.

Remark 17.63. Note that this shows $n \not\to (cn^{1/3})_2$ for some constant c. Compare this with the construction by Abbott, previously assigned as a Challenge problem, that showed $n \not\to cn^{\alpha}$ where $\alpha = \log_5 2 = 0.43\ldots$. — Nagy's construction was later generalized by Peter Frankl and Richard M. Wilson (1980) to a constructive proof that $n \not\to n^{\epsilon}$ for any constant $\epsilon > 0$.

BONUS+ 17.64. (8 points) A $(0,1)$-matrix is a matrix whose entries are $a_{ij} \in \{0,1\}$. A $k \times \ell$ *submatrix* is obtained by selecting an arbitrary set of k rows and ℓ columns and looking at the $k\ell$ cells at their intersection. A submatrix is *homogeneous* if all of its entries are the same.
Prove: every $n \times n$ $(0,1)$-matrix has a $k \times k$ homogeneous submatrix where $k \sim (1/2)\log_2 n$. (In an earlier version of this posting, I erroneously claimed $k = \lfloor (1/2)\log_2 n \rfloor$. In fact, k will be slightly smaller but still asymptotically equal to $(1/2)\log_2 n$.

HW+ 17.65. (8 points) Prove that for all sufficiently large n there exists an $n \times n$ $(0,1)$-matrix that has no homogeneous $k \times k$ submatrix for $k = \lceil 2\log_2 n \rceil$. Hint. Use Erdős's probabilistic method. State the size of the sample space you use. (The previous posting of this problem erroneously omitted the factor 2 before $\log_2 n$. **The due date for this problem is extended to Thursday, June 6, before class**.)

CH 17.66. (15 points) For infinitely many values of n, give a constructive proof that there exists an $n \times n$ $(0,1)$-matrix that has no homogeneous $k \times k$ submatrix for $k > \sqrt{n}$.