Graph Theory: CMSC 27530/37530 Lecture 18

Lecture by László Babai
Notes by Geoffrey West
Revised by instructor

May 30, 2019

Please remember that if you wish to receive a grade (either letter grade or P/F) then **you are expected to attend the last class, Thursday, June 6.** Most problems in this problem set are due Tuesday, June 4, but some of the problems are due on Thursday, June 6.

This problem set includes some not too difficult Challenge problems; you might try them until Thursday if you wish them to count toward your grade.

HW+ and **Bonus+** indicate homework and Bonus problems due Thursday, June 6.

DIRECTED GRAPHS

Definition 18.1. Let H be an orientation of the graph G. The **out-degree** $\deg_H^+(v)$ of vertex v in H is the number of edges of the form $v \to w$ (edges starting at v and directed away from v). The **in-degree** $\deg_H^-(v)$ is defined analogously. (We drop the subscript H if H is clear from the context.)

DO 18.2. Observe that $\deg_H^+(v) + \deg_H^-(v) = \deg_G(v)$.

DO 18.3 (Directed Handshake Theorem)**.** Prove: $m = \sum_{v \in V} \deg^+(v) = \sum_{v \in V} \deg^-(v)$, where m is the number of edges of G.

HW 18.4. (5+4 points) As before, let H be an orientation of the graph $G = (V, E)$. Assume $(\forall v \in V)(\deg_H^+(v) \leq k)$.

(a) Prove: $\chi(G) \leq 2k + 1$. State the coloring algorithm you use to prove this result. (Hint. Look further down in these notes.)

(b) Prove: the bound $2k+1$ is tight for every $k \geq 1$. (For every k you need to find a graph G and an orientation H of G such that $\chi(G) = 2k + 1$ while all outdegrees in H are $\leq k$.)

Definition 18.5. As before, let H be an orientation of the graph G. A **directed walk** from v to w in H is a walk in G that observes the orientation of the edges: $v \to u_1 \to \cdots \to w$. We say that w is **accessible** from v in H if there exists a $v \to \cdots \to w$ directed walk in H.

Definition 18.6. As before, let H be an orientation of the graph $G = (V, E)$. Let $s \neq t \in V$. An (s, t) **cut** is a partitition $V = A \sqcup B$ such that $s \in A$ and $t \in B$ and there is no edge oriented from A to B, i. e., if $\{v, w\} \in E$ and $v \in A$, $w \in B$ then the orientation in H is $w \to v$.

The next exercise describes a "good characterization" of accessibility.

HW 18.7. (7 points) As before, let H be an orientation of the graph $G = (V, E)$. Let $s \neq t \in V$. Prove: t is accessible from s if and only if there is no (s, t) cut.

Definition 18.8. We say that H is **strongly connected** if $(\forall v, w \in V)(w$ is accessible from $v)$.

HW 18.9. (2 points) Find the smallest connected graph that has no strongly connected orientation.

Definition 18.10. A **bridge** in a connected graph G is an edges e such that the graph $G - e$ is disconnected. Here $G - e$ is the graph G minus the edge e but the endpoints of e are retained, so $V(G - e) = V(G)$.

DO 18.11. An edge e in a connected graph G is a bridge if and only if e does not belong to any cycle in G.

CH 18.12. Prove: a connected graph G has a strongly connected orientation if and only if G has no bridge.

Definition 18.13. A **tournament** is an orientation of the complete graph.

CH 18.14. Prove: every strongly connected tournament with $n \geq 3$ vertices has a directed Hamilton cycle.

BONUS 18.15. (8 points) Prove: almost all tournaments are strongly connected. — Define the meaning of this statement.

Definition 18.16. As before, let H be an orientation of the graph $G = (V, E)$. We say that H is **acyclic** if there is no directed cycle in H. A **topological sort** of H is a numbering of the vertices, $v_1, \ldots, v_n$, such that if $v_i \to v_j$ is the orientation in H of the edge $\{i, j\} \in E$ then $i < j$ (all edges are oriented forward).

DO 18.17. Prove: H is acyclic if and only if H has a topological sort.

Remark 18.18. The definition of **directed graphs** permits two-way orientation of some of the edges, so oriented graphs are a proper subclass of directed graphs. However, most results stated in this section extend to directed graphs.

FIBONACCI NUMBERS
(DO solve problem DO 18.21. You may skip the rest of this section for now and return to it at your leisure.)

Notation 18.19. Let a, b be integers. We say that a **divides** b if $(\exists x \in \mathbb{Z})(b = ax)$. Notation: $a \mid b$. For instance, $6 \mid 18$ and $8 \nmid 18$.

DO 18.20. True or false: $0 \mid 0$?

DO 18.21. Let $f(t) = a_0 + a_1 t + \ldots + a_n t^n$, where $a_i \in \mathbb{Z}$ and $a_0, a_n \neq 0$. If f has a rational root r/s where $r, s \in \mathbb{Z}$, $s \neq 0$, and $\gcd(r, s) = 1$, then $r \mid a_0$ and $s \mid a_n$. In particular, if f is monic, then every rational root of f is an integer dividing a_0.

DO 18.22. ♡ Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.

(a) Determine the eigenvalues of A.

(b) Determine the entries of the matrix A^n.

(c) Prove, in half a line and without any further calculation, that $F_{n+1} + F_{n-1} = \phi^n + \overline{\phi}^n$ where ϕ is the golden ratio and $\overline{\phi}$ is the algebraic conjugate of ϕ, i. e., ϕ and $\overline{\phi}$ are the two roots of the polynomial $t^2 - t - 1$.

Here F_n is the n-th Fibonacci number. (Recall the numbering: $F_0 = 0, F_1 = 1$.)

DO 18.23. Prove: $F_{n+1}F_{n_1} - F_n^2 = (-1)^n$. Prove this with no calculation; use the preceding exercise, observing that $\det(A) = -1$.

DO 18.24. Prove: if $k \mid \ell$ then $F_k \mid F_\ell$, where F_k is the k-th Fibonacci number.

CH 18.25. (6 points) Prove: $\gcd(F_k, F_\ell) = F_d$ where $d = \gcd(k, \ell)$. If you use any of the the preceding exercises, prove them.

EXPONENTIAL FUNCTION OF A MATRIX
(This section is meant for your entertainment only. Feel free to skip it.)

Definition 18.26. Let $A \in M_n(\mathbb{C})$. Define e^A by

$$\mathrm{e}^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k . \tag{1}$$

DO 18.27. Prove that the sum (1) converges for every A. (Convergence is in any reasonable metric.)

DO 18.28. Prove: if $A \sim B$ (similar matrices) then $\mathrm{e}^A \sim \mathrm{e}^B$.

DO 18.29. Prove: if $\mathrm{spec}(A) = \{\{\lambda_1, \ldots, \lambda_n\}\}$ $(\lambda_i \in \mathbb{C})$ then $\mathrm{spec}(\mathrm{e}^A) = \{\{\mathrm{e}^{\lambda_1}, \ldots, \mathrm{e}^{\lambda_n}\}\}$.

DO 18.30. Let $A, B \in M_n(\mathbb{C})$. Consider the equation $\mathrm{e}^{A+B} = \mathrm{e}^A \mathrm{e}^B$. (a) Show that this equation does not always hold. (b) Prove that the equation does hold if A, B commute, i. e., $AB = BA$.

DO 18.31. Let $A \in M_n(\mathbb{R})$. Consider the function $f(t) = e^{tA}$ where $t \in \mathbb{R}$ (so $f : \mathbb{R} \to M_n(\mathbb{R})$). Prove:

$$\frac{d}{dt} f(t) = A \cdot f(t)\,. \tag{2}$$

This equation underlies the solution of systems of homogeneous linear ODEs.

CH 18.32. Let A be a real symmetric matrix and $B = e^{iA}$ where $i = \sqrt{-1}$. Let $C_n = I + B + B^2 + \cdots + B^n$. Prove: the sequence $\{C_n\}$ is bounded (in any reasonable metric).

DO 18.33. Let $f(t) = \sum_{k=0}^{\infty} a_k t^k$ be a power series ($a_i \in \mathbb{C}$) with convergence radius r. Let $A \in M_n(\mathbb{C})$. Define $f(A)$ analogously to Def. 18.26.
Assume $|\lambda| < r$ for every $\lambda \in \operatorname{spec}(A)$. Prove: the series $f(A)$ converges.

MULTIPLICITY OF EIGENVALUES: ALGEBRAIC VS. GEOMETRIC

Definition 18.34. Let $A \in M_n(\mathbb{C})$. The **algebraic multiplicity** of an eigenvalue λ is its multiplicity in the characteristic polynomial. We denote this number $\operatorname{alg}_A(\lambda)$.

Definition 18.35. The **eigenspace** for an eigenvalue λ is the set

$$U_\lambda(A) = \{\mathbf{x} \in \mathbb{C}^n \mid A\mathbf{x} = \lambda\mathbf{x}\} \leq \mathbb{C}^n\,. \tag{3}$$

So the eigenspace $U_\lambda(A)$ consists of the eigenvectors of A to eigenvalue λ plus the zero vector.

DO 18.36. Prove: $U_\lambda(A)$ is a subspace of $\mathbb{C}^n$.

DO 18.37. $U_\lambda(A) = \ker(\lambda I - A)$.

Definition 18.38. The **geometric multiplicity** of λ is the maximum number of linearly independent eigenvectors to λ in $\mathbb{C}^n$. We denote this number $\operatorname{geom}_A(\lambda)$.

DO 18.39. Prove:

(a) $\operatorname{geom}_A(\lambda) = \dim(U_\lambda(A))$

(b) $\operatorname{geom}_A(\lambda) = n - \operatorname{rk}(\lambda I - A)$. (Hint. Rank–Nullity)

DO 18.40. Let $A, B \in M_n(\mathbb{C})$ be similar matrices. Prove, for all $\lambda \in \mathbb{C}$, that

(a) $\operatorname{alg}_A(\lambda) = \operatorname{alg}_B(\lambda)$

(b) $\operatorname{geom}_A(\lambda) = \operatorname{geom}_B(\lambda)$

DO 18.41. Show that the geometric and the algebraic multiplicities are not always equal.

Solution. Consider the matrix

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Then $f_A(t) = t^2$, so $\operatorname{alg}_A(0) = 2$. However, $\operatorname{rk}(A) = 1$, so by part (b) of Exercise DO 18.39, $\operatorname{geom}(0) = 2 - 1 = 1$. $\square$

DO 18.42. Consider the matrix $A \in M_n(\mathbb{R})$ defined as

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

Show the following.

(a) $\text{alg}(0) = n$.

(b) $\text{geom}(0) = 1$.

DO 18.43.

$$\text{geom}_A(\lambda) \leq \text{alg}_A(\lambda).$$

Hint. Let $\mathbf{u}_1, \dots, \mathbf{u}_k$ be a basis of $U_\lambda(A)$. Extend this list to a basis of $\mathbb{C}^n$. Switch to this basis; the action of A on $\mathbb{C}^n$ will now be described by a matrix of the form

$$B = \left[\begin{array}{c|c} \lambda I & * \\ \hline 0 & * \end{array}\right]$$

where I is the $k \times k$ identity matrix. Change of basis results in a similar matrix, so $A \sim B$ and therefore $f_A = f_B$.

DO 18.44. A matrix $A \in M_n(\mathbb{C})$ is diagonalizable (over $\mathbb{C}$) if and only if

$$(\forall \lambda)(\text{geom}_A(\lambda) = \text{alg}_A(\lambda)) . \tag{4}$$

DO 18.45. A matrix $A \in M_n(\mathbb{R})$ is diagonalizable over $\mathbb{R}$ if and only if f_A is real-rooted and Eq. (4) holds for all $\lambda \in \mathbb{R}$.

DO 18.46. If A is a symmetric real matrix then Eq. (4) holds for all $\lambda \in \mathbb{R}$.

SPECTRUM VS. MAX DEGREE

DO 18.47. Let G be a graph with vertex set $[n]$ and let A_G be its adjacency matrix. Let $\mathbf{x} = (x_1, \dots, x_n)^T \in \mathbb{R}^n$. Let $A_G\mathbf{x} = \mathbf{y} = (y_1, \dots, y_n) \in \mathbb{R}^n$. Recall that

$$y_i = \sum_{j: j \sim i} x_j . \tag{5}$$

Consider the "star" graph $G = K_{1,n-1}$. It has the adjacency matrix

$$A_G = \left(\begin{array}{c|cccc} 0 & 1 & 1 & \dots & 1 \\ \hline 1 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \dots & 0 \end{array}\right)$$

What is $\lambda_1(G)$? Since G is connected, $\lambda_1(G)$ admits an all-positive eigenvector $\mathbf{x} = (x_1, \ldots, x_n)^T$. Then, by Eq. (5),

$$\lambda x_1 = \sum_{i=2}^{n} x_i \,.$$

For $i \geq 2$, Eq. (5) gives $\lambda x_i = x_1$. We thus have

$$\lambda^2 x_1 = \sum_{i=2}^{n} \lambda x_i = \sum_{i=2}^{n} x_1 = (n-1)x_1$$

so $\lambda = \sqrt{n-1}$.

DO 18.48. The spectrum of $K_{1,n-1}$ is $\{\{\sqrt{n-1}, 0^{n-2}, -\sqrt{n-1}\}\}$. (This notation means that the multiplicity of 0 is $n-2$.)

Solution. Note that $\text{rk}(A_G) = 2$. (Why?) Therefore, by part (b) of Exercise DO 18.39, the multiplicity of 0 is $n-2$. (Why?) The remaining eigenvalue can only be $-\lambda_1$ because the trace is zero. (Another reason: $\lambda_n = -\lambda_1$ because G is bipartite.) □

DO 18.49. $(\forall G)(\lambda_1(G) \geq \sqrt{\Delta(G)})$, where $\Delta(G) = \deg_{\max}(G)$.

Proof. $G \supseteq K_{1,\Delta}$ and therefore $\lambda_1(G) \geq \lambda_1(K_{1,\Delta})$. □

HW 18.50. (6 points) Find $\text{spec}(K_{r,s})$. Find an eigenvector to $\lambda_1(K_{r,s})$.

BONUS 18.51. (5 points) If G is d-regular and $\text{diam}(G) \geq 4$, then $\lambda_2 \geq \sqrt{d}$.

CH 18.52. (6 points) If G is d-regular ($d \geq 2$) and $n > kd^3$, then $\lambda_k \geq \sqrt{d}$.

Theorem 18.53 (Godsil)**.** *If T is a tree, then $\lambda_1(T) \leq 2\sqrt{\Delta - 1}$.*

For the proof, we need the following lemma.

DO 18.54. Let $A = (a_{ij}) \in M_m(\mathbb{R})$ be a non-negative matrix ($(\forall i,j)(a_{ij} \geq 0)$). Let $b_i = \sum_j a_{ij}$ be the i-th row sum. Then

$$(\forall \lambda \in \text{spec}(A))(|\lambda| \leq b_{\max}) \,. \tag{6}$$

The proof of this lemma is analogous to the proof that $\lambda_1(G) \leq \Delta(G)$.

Proof of Theorem 18.53. Let $A = A_T$ be the adjacency matrix. Choose a vertex to be the "root." For any $v \in V$, let $h(v)$ be the length of the unique path between v and the root (the "height" of v). Let D be the diagonal matrix

$$D = \text{diag}\left(\sqrt{\Delta - 1}^{\,h(v)}\right)$$

Consider the matrix $B = DAD^{-1}$. Since $B \sim A$, they have the same spectrum. By the Lemma (DO 18.54) we can bound $\lambda_1(T)$ by the maximum row-sum of B. Let $v \in V$, other than the root. Let w be the neighbor of v along the unique path from v to the root. The row corresponding to v in B has value $\frac{1}{\sqrt{\Delta-1}}$ in each column corresponding to a neighbor of v, except for w. In the column corresponding to w, the matrix has entry $\sqrt{\Delta-1}$. So the row sum is

$$\sqrt{\Delta - 1} + (\deg(v) - 1) \cdot \frac{1}{\sqrt{\Delta - 1}} \leq 2\sqrt{\Delta - 1}\,.$$

□

Notation 18.55 (Rooted tree)**.** Let T be a tree and r a vertex designated as the "root." The **parent** of a vertex $p \neq r$ is the neighbor of p along the unique $p - \cdots - r$ path. Let p' denote the parent of p. Vertex q is a **child** of p if $p = q'$. An iterated parent is an **ancestor**. The parent function determines the rooted tree; vertices p and q are adjacent exactly if $p = q'$ or $q = p'$.

DO 18.56. A function $f : V \setminus \{r\} \to V$ can be used as the parent function to define a rooted tree, rooted at r, if and only if no vertex is its own ancestor.

Definition 18.57 (Tree of paths)**.** Let G be a connected graph with a root vertex r. For a path $P \subseteq G$ of length ≥ 1 from the root, let P' denote the path $P - p$, where $p \neq r$ is the other endpoint of P. We define the rooted tree T_G, called the "tree of paths" for (G, r), as follows. The vertices of T_G are the paths in G from r, including the path $\{r\}$ of length zero which will be the root of T_G. We declare P' to be the parent of P (if P is not the root of T_G).

DO 18.58. Prove that T_G is a tree.

DO 18.59. Prove that $\Delta(T_G) = \Delta(G)$.

Heilmann and Lieb gave three proofs of their theorem that the roots of the matchings polynomial are real. Godsil gave a fourth proof.

Theorem 18.60 (Godsil)**.** *If G is a connected graph then $\mu_G \mid f_{T(G)}$ (where μ_G is the matchings polynomial).*

It follows that all roots of μ_G are eigenvalues of T_G and therefore they are real. Moreover, we also get a bound on the roots.

Corollary 18.61. *Let G be a graph and λ a root of its matchings polynomial. Then $|\lambda| \leq 2\sqrt{\Delta(G) - 1}$.*

This result holds regardless of whether G is connected. The extention from connected to disconnected graphs follow from the following observation.

DO 18.62. Let $G = K \sqcup L$ (disjoint union of the graphs K and L). Then $\mu_G = \mu_k \cdot \mu_L$.

WILF'S EIGENVALUE BOUND ON THE CHROMATIC NUMBER

A previous HW asked to show $\chi(G) \leq 1 + \lambda_1(G)$ (Herbert Wilf, 1961). The proof is via the "smart-greedy" algorithm. The algorithm does not mention eigenvalues; the bound comes from the analysis of the algorithm.

The algorithm recursively constructs a coloring $c : V \to \mathbb{N}$.

```
Procedure SMART-GREEDY(G):
    if (n = 1):
        return c(v) = 1
    else:
        Pick a vertex v of minimum degree
        c ← SMART-GREEDY(G − v)
        c(v) ← first available color
        return c
```

Claim 18.63. The algorithm uses at most $1 + \lambda_1(G)$ colors.

Proof. The inductive hypothesis is that smart-color$(G-v)$ uses at most $1+\lambda_1(G-v)$ colors. Since $\lambda_1(G-v) \leq \lambda_1(G)$, it follows that smart-color$(G-v)$ uses at most $1+\lambda_1(G)$ colors. Now $\deg(v) \leq$ average degree $\leq \lambda_1(G)$, so among the $\lfloor 1 + \lambda_1(G) \rfloor$ colors, there is at least one "free color" for v to use, and thus G requires at most $1 + \lambda_1(G)$ colors. This completes the inductive step. The base case is $G = K_1$, $\lambda_1(K_1) = 0$, and the algorithm uses 1 color to color K_1. □

The same algorithm can be analyzed in other ways to obtain other bounds on the chromatic number.

Fact 18.64. *Every planar graph has a vertex of degree* ≤ 5.

DO 18.65. Use Fact 18.64 to prove that every planar graph is 6-colorable.

TRIANGLE-FREE GRAPHS WITH LARGE CHROMATIC NUMBER

If $G \supseteq K_5$ then $\chi(G) \geq 5$. The mistake made in the first attempted proof of the Four-color conjecture was the assumption that the converse also holds: by proving that K_5 is not planar, the person believed to have proved that planar graphs are 4-colorable. This inference is wrong; in fact, triangle-free graphs can have arbitrarily large chromatic number. It was an exercise in this class to show that there exists a triangle-free graph that is not 3-colorable. The hint said such a graph with 11 vertices and a rotational symmetry of order 5 can be constructed; the resulting graph is called Grötzsch's Graph, after Herbert Grötzsch who published this graph in 1959. In fact, years earlier Jan Mycielski proved a stronger result (1955): he constructed a sequence of triangle free graphs with increasing chromatic number. We describe his inductive construction.

The sequence starts with $M_2 := K_2$. For any k, we define M_{k+1} by the "doubling plus one" method. Let $v_1, \ldots, v_r$ be the vertices of M_k. The vertex set of M_{k+1} is

$$V(M_{k+1}) = \{v_0, \ldots, v_r\} \cup \{u_0, \ldots, u_r\} \cup \{w\}.$$

So $|V(M_{k+1})| = 2 \cdot |V(M_k)| + 1$. The edge set of M_{k+1} is defined by the following relations.

1. The edges and non-edges of M_k are retained, so M_k is an induced subgraph of M_{k+1}:

$$v_i \sim_{M_{k+1}} v_j \iff v_i \sim_{M_k} v_j$$

2. Each u_i inheirits the neighborhood of v_i:

$$u_i \sim_{M_{k+1}} v_j \iff v_i \sim_{M_k} v_j$$

3. There are no edges among the u_i .

4. Vertex w is adjacent to each u_i and none of the v_i .

Figure 1 show this process applied to C_5 (which is M_3) to obtain M_4. So $M_2 = K_2,\ M_3 = C_5$,

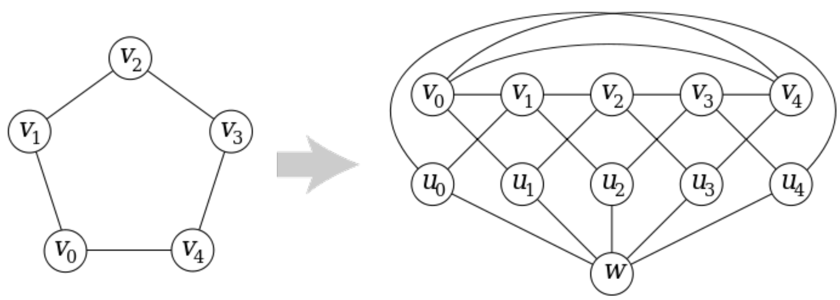

Figure 1: The "doubling plus one" method on C_5.

M_4 is Grötzsch's graph.

DO 18.66. $\chi(M_k) = k$ and $M_k \not\supset K_3$.

Corollary 18.67 (Mycielski)**.** *There exist triangle-free graphs of arbitrarily large chormatic number.*

The following lemma will be helpful for the proof of exercise DO 18.66.

HW 18.68. (5 points) Let $G = (V, E)$ and assume $\chi(G) = k$. Let $f : V \to [k]$ be a k-coloring of G. Then for each color $i \in [k]$ there exists a vertex $v \in V$ such that $f(v) = i$ and every color other than i occurs among the colors of the neighbors of v.

How large are Mycielski's graphs?

DO 18.69. Prove: M_k has order $3 \cdot 2^{k-1} - 1$.

So these graphs grow at an exponential rte as a function of their chromatic number. Erdős showed (1959) that in fact mauch smaller triangle-free graphs exist with a given chromatic number; they only need to grow at a polynomial rate ($n = k^{3+\epsilon}$).

Theorem 18.70 (Erdős, 1959)**.** *For any $\epsilon > 0$, for all sufficiently large k there exists a triangle-free graph of chromatic number k with at most $k^{3+\epsilon}$ vertices.*

The proof is based on random choice of graphs according to a distribution called the Erdős-Rényi random graph model $\mathbb{G}(n, p)$ after a pair of articles by Paul Erdős and Alfréd Rényi (1960/61) in which they gave a detailed analysis of the evolution of their random graphs as a function of the edge probability parameter p.

Definition 18.71. Let Ω_n denote the set of all graphs on vertex set $[n]$; so $|\Omega_n| = 2^{\binom{n}{2}}$. The **Erdős-Rényi random graph model** $\mathbb{G}(n,p)$ is a distribution on Ω_n. We perform $\binom{n}{2}$ indepedent Bernoulli trials with probability p of success, one trial B_e for each edge e of the complete graph K_n. In case B_e succeeds, the edge e is included in our random graph $\mathcal{G}$, otherwise not. So the probability that a given graph G on vertex set $[n]$ with m edges is the outcome of our experiment is

$$P(\mathcal{G} = G) = p^m(1-p)^{\binom{n}{2}-m} . \tag{7}$$

DO 18.72. Let v be a vertex of a $\mathbb{G}(n,p)$ random graph. Then $E(\deg(v) = (n-1)p$.

DO 18.73. The expected number of triangles in a $\mathbb{G}(n,p)$ random graph is $\binom{n}{3}p^3 \sim (np)^3/6$.

In fact, the degre of a given vertex is binomially distributed as a sum of $n-1$ independent Bernoulli trials with probability p of success.
The following estimate is useful.

DO 18.74. For all $x \in \mathbb{R}$, $\quad 1 + x \leq \mathrm{e}^x$.

Proof of Theorem 18.70. Let us fix some small positive number θ, its value to be determined later. Let $p := n^{\theta-1}$, and let $\mathcal{G}$ be random graph from the distribution $G(n,p)$. So the expected degree of a vertex is $(n-1)\cdot n^{\theta-1} \sim n^\theta$. Our lower bound on the chromatic number will be achived by giving an upper bound on the independence number, using the inequality

$$\chi(G) \geq \frac{n}{\alpha(G)}.$$

Ideally we would want to set the value of θ so that the following two events both have high probability:

(a) $\mathcal{G}$ has no triangles;

(b) $n/\alpha(\mathcal{G})/n \geq k$.

Unfortunately this is impossible. According to Exercise DO 18.73, the expected number of triangles in $\mathbb{G}$ is close to $(np)^3/6 = n^{3\theta}/6$, so to have a good chance of having no trinagles, we would need $\theta \leq 0$, but that would make the chromatic number at most three with high probability. So we need to compromise on item 1. Here is the modified plan. We need to find a (not too large) graph G such that

(A) G has at most $n/2$ triangles;

(B) $n/\alpha(G) \geq 2k$.

First let us prove that if we found a graph that has both properties then we can construct another graph G' that is triangle-free and has chromatic number $\geq k$.

Pick a vertex from each triangle of G and remove it. This will leave a graph G' with at least $n/2$ vertices, no triangles, and $\alpha(G') \leq \alpha(G)$ (because G' is an induced subgraph of G). Therefore

$$\chi(G') \geq \frac{n/2}{\alpha(G')} \geq \frac{1}{2}\frac{n}{\alpha(G)} \geq k . \tag{8}$$

So we need to find a (not too large) G that satisfies (A) and (B). We shall calibrate θ so that our random graph $\mathcal{G}$ will satisfy both (A) and (B) with probability greater than 1/2 and therefore both of them simultaneously with positive probability, proving the existence of a graph G that satisfies both conditions.

Let $t(G)$ denote the number of triangles of the graph G. Our first goal is that $E(t(\mathcal{G})) \leq n/6$. If we achieved this, by Markov's inequality we shall have

$$P(t(\mathcal{G}) \geq n/2) \leq 1/3\,. \tag{9}$$

Since $E(t(\mathcal{G})) < n^{3\theta}/6$, this goal will be reached assuming $n^{3\theta} \leq n$, i. e.,

$$\theta \leq 1/3\,. \tag{10}$$

Next we estimate $\alpha(\mathcal{G})$. Let us fix a set $A \subseteq [n]$ of size $|A| = a$.

$$P(A \text{ is independent }) = (1-p)^{\binom{a}{2}} < \mathrm{e}^{-p\binom{a}{2}}\,.$$

(We used Exercise DO 18.74.) Trying every subset of size a, by the union bound we obtain

$$P(\alpha(\mathcal{G}) \geq a) < \binom{n}{a}\mathrm{e}^{-p\binom{a}{2}} < \frac{1}{a!}\left(n\mathrm{e}^{-p(a-1)/2}\right)^a \leq \frac{1}{a!} \tag{11}$$

assuming $n \leq \mathrm{e}^{p(a-1)/2}$, which means, taking logarithms, that $\ln n \leq p(a-1)/2$, i. e., $p \geq (2\ln n)/(a-1)$.

Our goal is that $a-1 \leq n/(2k)$. To achive this, we need $p \geq (2\ln n)(2k/n)$, i. e., $n^\theta \geq 4k\ln n$, i. e.,

$$\theta \geq \frac{\ln(4k) + \ln\ln n}{\ln n}\,. \tag{12}$$

So, combining the two constraints on θ, inequalities (10) and (12), we need θ to satisfy

$$\frac{\ln(4k) + \ln\ln n}{\ln n} \leq \theta \leq \frac{1}{3}\,. \tag{13}$$

For such θ to exist, it is necessary and sufficient that

$$\frac{\ln(4k) + \ln\ln n}{\ln n} \leq \frac{1}{3}\,. \tag{14}$$

Since the left-hand side approaches zero as $n \to \infty$ for fixed k, this inequality indeed holds for all sufficiently large n.

Our last question is, how large is sufficiently large as a function of k.

Claim 18.75. For any constant $\epsilon > 0$, inequality (14) holds for all sufficiently large k, assuming $n \geq k^{3+\epsilon}$. (The threshold for k depends on ϵ.)

Indeed, $\ln\ln n/\ln n \to 0$ as $k \to \infty$ (and with it, $n \to \infty$), and $(\ln 4 + \ln k)/((3+\epsilon)\ln k) \to 1/(3+\epsilon)$, so for sufficiently large k these two quantities add up to less than 1/3. This proves the Claim and with it, completes the proof of Theorem 18.70. □

But Erdős's result is about much more than triangle-free graphs. Spectacularly, he was able to eliminate all short cycles and still claim large chromatic number.

Theorem 18.76 (Erdős)**.** *There exist graphs of arbitrarily large girth and chromatic number.*

If you like formal expressions, here is the theorem.

$$(\forall g)(\forall k)(\exists G)(\chi(G) \geq k \wedge \text{girth}(G) \geq g).$$

The proof is, *mutatis mutandis,* the same as the the proof above; instead of $k^{3+\epsilon}$, the bound on n will be $k^{g+\epsilon}$.

What does large girth mean?

DO 18.77. Assume G has girth $\geq g$. Let v be a vertex and $B(v)$ be the set of vertices at distance $\leq (g-1)/2$ of v (the "ball of radius $(g-1)/2$ about v"). Then the subgraph induced by $B(v)$ is a tree.

So the graph is locally a tree, yet it has large chromatic number. Why is the chromatic number large? Theorem 18.76 is the first in a long series of results by Erdős that indicate that *there is no local explanation for high chromatic number.* Erdős, more than anyone else among his contemporaries, recognized that the chromatic number is a deep structural parameter of the graph. Theorem 18.76 had tremendous influence on combinatorics as well as on the theory of computing where the complexity of computing or approximating the chromatic number became a central subject. Here is the most remarkable complexity theoretic result about the chromatic number.

Theorem 18.78 (Johan Håstad, 2000)**.** *The following holds for any $0 < \epsilon < 1/2$. I give you two graphs. I promise that one of them is colorable by n^ϵ colors, the other requires at least $n^{1-\epsilon}$ colors. If you can tell, in polynomial time, which is which, then $P = NP$. (In particular, in that case, the exact chromatic number can be computed in polynomial time.)*

The proof is built on the theory of "interactive proofs" and makes extensive use of Fourier analysis on the Bolean cube.

Håstad also proved that the **exact same result holds for the independence number** (and therefore for the clique number).

EXPLICIT CONSTRUCTION: KNESER'S GRAPHS

Erdős proves the existence of graphs of large girth and chromatic number but does not show how to construct them. This non-constructive nature is characteristic for proofs of existence via the probabilistic method. Much effort has been devoted to explicit cnstructions. Somewhat surprisingly, avoiding 4-cycles seems far more difficult than avoiding odd cycles. Here is a class of graphs of large chromatic number, without short odd cyles.

Definition 18.79 (Kneser's graphs)**.** Let $s \geq 1$ and $r \geq 2s+1$. The Kneser graph $Kn(r,s)$ has $\binom{r}{s}$ vertices, labeled by the s-subsets of $[r]$. Two vertices v_A and v_B are adjacent if A and B are disjoint ($A, B \in \binom{[r]}{s}$).

DO 18.80. $Kn(5, 2)$ is isomorphic to the Petersen graph.

HW+ 18.81. (8 points) Prove: $\chi(Kn(r, s)) \le r - 2s + 2$.

Martin Kneser was the first to study these graphs and he conjectured that their chromatic number is exactly $r - 2s + 2$. This problem remained open for more than two decades, until Lovász confirmed it in 1978 using a rather surprising topological method. Lovász introduced the *neighborhood complex* of a graph – a simplicial complex where the sets $N_G(v)$ are the maximal simplices, and proved, using the Borsuk–Ulam Theorem, that if this complex has high topological connectivity then G has high chromatic number. Almost immediately (1978), Imre Bárány gave a more elementary proof, using convex geometry to reduce Kneser's conjecture to the Borsuk–Ulam theorem.

Theorem 18.82 (Lovász, 1978)**.** $\chi(Kn(r, s)) = r - 2s + 2$.

So if $r - 2s$ is large then the Kneser graph has high chromatic number. Let us now study the short cycles of the Kneser graph.

HW+ 18.83. (5 points) Prove: If $r \ge 2s + 2$ then $Kn(r, s)$ contains a 4-cycle.

In fact, when $r-2s$ is large (the case of interest to us), then $Kn(r, s)$ contains tons of 4-cycles.

HW+ 18.84. (5 points) $Kn(r, s)$ contains the complete bipartite graph $K_{L,M}$ where $L = \binom{\lceil r/2 \rceil}{s}$ and $M = \binom{\lfloor r/2 \rfloor}{s}$.

However, if $2s/r$ is close to 1 then the Kneser's graph has no short odd cycles.

HW+ 18.85. (6 points) Prove: $Kn(r, s)$ has no odd cycles shorter than $r/(r - 2s)$.

Corollary 18.86. *Let $r = k^2$ and $s = k(k-1)/2$. Then $K(r, s)$ has chromatic number $k+2$ and no odd cycle shorter than k.*

Explicit constructions of graphs of large girth and large chromatic number only came with the advent of *Ramanujan graphs* (Margulis, Lubotzky–Phillips–Sarnak, 1988).

INFINITE GRAPHS
(For your entertainment only; feel free to ignore this section.)

Legal coloring has the same meaning for inifinite graphs as for finite graphs, but the "number" of colors may be an infinite cardinal such as $\aleph_{\omega+2}$. Any non-empty set of cardinals has a minimum, so the notion of chromatic number makes sense.

Infinite graphs sometimes have finite chromatic number. A fundamental result by Erdős and De Bruijn asserts that this depends on their finite subgraphs only.

CH 18.87 (Erdős–De Btuijn, 1949)**.** Let k be a positive integer. An inifinite graph G is k-colorable if and only if all finite subgraphs of G are k-colorable.

Give three proofs of this theorem: (a) using Gödel's compactness theorem in first-order logic, (b) using Tykhonoff's compactness theorem in topogy, and (c) a direct proof using Zorn's lemma.

Some constructions of high-chromatic graphs without short odd cycles generalize to infinite chromatic numbers.

CH 18.88. For any positive integer k and any infinite cardinal $\mathfrak{m}$ there exists a graph G such that G has chromatic number $\geq \mathfrak{m}$ and G has no odd cycles of length less than k.

On the other hand, 4-cycles cannot be avoided. Erdős and András Hajnal have shown that a graph with uncountable chromatic number necessarily contains 4-cycles.

CH 18.89 (Erdős–Hajnal)**.** If a graph G has uncountable chromatic number then G contains a 4-cycle. In fact, G contains a complete bipartite graph $K_{m,\aleph_1}$ for every positive integer m.