Honors Algorithms CMSC-27230 Third Quiz. March 5, 2020

Instructor: László Babai

NAME (print with LARGE letters) ____

Are you using a cheat sheet? Circle the answer: YES NO
If YES, do not forget to **hand it in** at the end of the test.

Please DO NOT SIT next to anyone (leave an empy seat). Please do not sit near a person to whom you usually sit near in class.

SHOW ALL YOUR WORK. Do NOT use book, notes, or scrap paper. You may use a "cheat sheet": one page of HANDWRITTEN notes in English, written in ink in your handwriting with your name printed in large English letters on the top. NO PHOTOCOPIES! Hand in your cheat sheet with the test.

The use of ELECTRONIC DEVICES is STRICTLY FORBIDDEN.

Write your answers IN THE SPACE PROVIDED. You may **continue on the reverse.** DO NOT USE paper other than the problem sheet provided.

When describing an algorithm in pseudocode, **explain the meaning of your variables** (in English).

This quiz contributes 6% to your course grade.

1. (7+5+5 points) Consider an algorithm that consists of a single **while** loop. The algorithm takes as input a pair (x, y) of positive integer variables. The internal variables of the algorithm (the variables that may be updated in the course of the algorithm) are $u, v, w_1, \ldots, w_k$. The variables $w_1, \ldots, w_k$ are Boolean (take value from $\{0, 1\}$); the variable u is a residue class modulo x and the variable v is a y-bit integer, where initial zeros are permitted. Determine

   (a) the size of configuration space $\mathcal{C}$

   (b) the number of predicates over the configuration space

   (c) the number of transformations of the configuration space

   Your answers should be simple formulas; no proof required.

2. (6+6+6 points) Recall that in Dijkstra's algorithm, the input is a rooted, edge-weighted digraph (V, E, s, w) where $s \in V$ is the root and $w : E \to \mathbb{R}^+$ is the weight function. Here $\mathbb{R}^+$ denotes the set of nonnegative real numbers. The internal variables include, for each $v \in V$,

   - a variable called status(v) with values from {WHITE, GREY, BLACK},
   - a variable called parent(v) which takes values from $V \cup \{\text{NIL}\}$, and
   - a variable $t(v)$ that takes values from $\mathbb{R}^+ \cup \{\infty\}$. (The intended meaning of $t(v)$ is the total weight of the current min-weight $s \to v$ path.)

   For each statement below, decide whether the statement is a loop-invariant for Dijkstra's algorithm.

   (a) "if status(v) = white then parent(v) = NIL" YES NO

   (b) "if status(v) = black and $v \neq s$ then parent$(v) \neq$ NIL" YES NO

   (c) "the value of $t(v)$ cannot increase" YES NO

   Circle your answers, and describe the reason of each answer.

3. (5 points) Does multiplication of positive integers belong to the class $\mathcal{P}$ of computational problems? YES NO
   Circle your answer.
   Describe the reason for your answer.

4. (7 points) Recall the definition of NP: A language $L \subseteq \Sigma^*$ belongs to NP if

$$(\exists L_1 \in \mathcal{P})(\exists C, \Sigma_1)(\forall x \in \Sigma^*)(x \in L \iff (\exists w \in \Sigma_1^*)((x, w) \in L_1 \wedge |w| \leq |x|^C))$$

On a test, Fred mistakenly writes $\Longrightarrow$ instead of $\Longleftrightarrow$ in this definition. Fred's definition defines another language class, Fred-NP. Determine the class Fred-NP. Give a simple description of this class. Prove your answer.

5. (3+3+7 points)

   (a) Define the language corresponding to the decision version of the maximum independent set problem for graphs.
   Call this language IND-DEC.

   (b) Define the language corresponding to the decision version of the maximum clique problem for graphs.
   Call this language CLIQUE-DEC.

   (c) Given that IND-DEC is NP-complete, prove that CLIQUE-DEC is NP-complete.

6. (Bonus 7+7 points)

   (a) Given that 3-COL is NP-complete, prove that 4-COL is NP-complete. (k-COL denotes the set of k-colorable graphs.)

   (b) A Hamilton path in a graph is a path that passes through each vertex exactly once. (So this is a path of length $n-1$ where n is the number of vertices.) A Hamilton cycle in a graph is a cycle that passes through each vertex exactly once. (So this is a cycle of length n.) Let HAMPATH denote the set of graphs with a Hamilton path, and HAMCYCLE the set of graphs with a Hamilton cycle.

   THEOREM (Richard M. Karp, 1972)   HAMPATH is NP-complete.

   Do not prove this theorem. Use this theorem to prove that HAMCYCLE is NP-complete. (Of course, Karp proved this, too.)

(Note: Part (b) has been updated as announced at the beginning of the test.)