Solution to problem 13.210 (assignment HW7c)

13.210. What is the expected number of Aces in a poker hand? Show all your work. State the size of your sample space. Make sure you give a clear definition of the probability space and the random variables you introduce; the definition of your random variables will account for half the credit.

The answer is **5/13**, the number of cards in a poker hand times the proportion of Aces among the cards. We give two proofs of this simple and plausible answer. Both proofs will give intuitive explanations of the answer.

First proof. Let Ω_1 denote the set of 52 cards in the standard deck, so $|\Omega_1| = 52$. Let Ω_2 denote the set of poker hands, so $|\Omega_2| = \binom{52}{5}$.

Let Ω denote the sample space for this problem X the random variable that counts the Aces in the poker hand. So X is a function with domain Ω. This immediately rules out the naive answer that $\Omega := \Omega_1$. The natural answer is $\Omega := \Omega_2 :=$ since X is a function of the poker hand and now each poker hand is an elementary event. So now $|\Omega| = \binom{52}{5}$. Even this answer will be problematic, as we shall see below. In any case, for the definition of X we need to take the uniform distribution on Ω_2.

X counts, so we want to write X as a sum of indicator variables. What does X count? It counts the cards in the poker hand that are Aces. So, for $i \in [5]$, let Y_i denote the indicator of the event that the i-th card is the poker hand is an Ace. Now we have

$$X = \sum_{i=1}^{5} Y_i \,. \tag{1}$$

The trouble is, the random variable Y_i cannot be defined on the sample space Ω_2: each element of Ω_2 is an undordered set of 5 cards, there is no such thing as the first card, etc.

So we need to update our sample space so that each element of the sample space defines a first card, a second card, etc. Let $\Omega_3 :=$ the set of ordered 5-tuples of cards. If we want our variables Y_i to make sense, we need to choose $\Omega := \Omega_3$ (with the uniform distribution). So now $|\Omega| = |\Omega_3| = 52 \cdot 51 \cdot 50 \cdot 49 \cdot 48$. This choice should be the first sentence of the solution; after this, we can define Y_i and state Equation (1).

Only after this preparation can we apply the linearity of expectation to Equation (1), with the result that

$$E(X) = \sum_{i=1}^{5} E(Y_i) = \sum_{i=1}^{5} P(Y_i = 1)\,. \tag{2}$$

The event "$Y_i = 1$" is the event that the i card is an Ace. Since evey card has an equal chance of being the i-th card, and there are 4 Aces, the probability of this event is $4/52 = 1/13$. So our conclusion is

$$E(X) = \sum_{i=1}^{5} \frac{1}{13} = \frac{5}{13}\,. \tag{3}$$

We still have a technical step to do.

We need to justify the move from sample space Ω_2 (where X is naturally defined) to Ω_3, where technically we have a different variable, X', that has the same intuitive meaning (the number of Aces in yhe hand) but is a different function because it has a different domain.

We need to show that $E(X) = E(X')$. We notice that what we did was we blew up each elementary event in Ω_2 to $5! = 120$ elementary events in Ω_3 without changing the value of X.

Formally, for each $a \in \Omega_2$ (an unordered poker hand), let B_a denote the set of orderings of a. So $|B_a| = 5! = 120$, and the sets B_a partition Ω_3. Moreover, the value of X' is the same on all these 120 elementary events, namely, $(\forall a \in \Omega_2)(\forall b \in B_a)(X'(b) = X(a))$.

Therefore

(4)

$$E(X') = \sum_{b\in\Omega_3} \frac{X'(b)}{|\Omega_3|} = \sum_{a\in\Omega_2}\sum_{b\in B_a} \frac{X(b)}{|\Omega_3|} = \sum_{a\in\Omega_2} X(a)\cdot\frac{|B_a|}{|\Omega_3|} == \sum_{a\in\Omega_2} \frac{X(a)}{|\Omega_2|} = E(X)\,.$$

This completes the first proof. QED

The second proof (next page) will avoid this technical issue by choosing a different set of indicator variables.

Second proof. For this proof we use the sample space Ω_2 (of size $\binom{52}{5}$) (with the uniform distribution).

Let S denote the set of four suits, $S = \{\spadesuit, \heartsuit, \diamondsuit, \clubsuit\}$. For $s \in S$, let Z_s denote the indicator of the event that the Ace of suit s is in the poker hand. So, for instance, $Z_\spadesuit$ indicates the event that the Ace of spades is in the hand. Note that the random variables Z_s are defined on the domain Ω_2. We again have

$$X = \sum_{s \in S} Z_s \,. \tag{5}$$

Therefore

$$E(X) = \sum_{s \in S} E(Z_s) = \sum_{s \in S} P(Z_s = 1)\,. \tag{6}$$

The event "$Z_s = 1$" is the event that the Ace of suit s is in the hand. There are $\binom{51}{4}$ poker hands that include the Ace of suit s (why?), so the probability in question is

$$P(Z_s = 1) = \frac{\binom{51}{4}}{\binom{52}{5}} = \frac{5!}{4!} \cdot \frac{51 \cdots 48}{52 \cdot 51 \cdots 48} = \frac{5}{52}\,. \tag{7}$$

The conclusion is that

$$E(X) = \sum_{s \in S} P(Z_s = 1) = \sum_{s \in S} \frac{5}{52} = 4 \cdot \frac{5}{52} = \frac{5}{13}\,. \tag{8}$$

QED

Remark. The second solution involved a little calculation (Eq. (7)) that detracts from its intuitive nature. By moving to the sample space Ω_3 we can avoid even this calculation. For $i \in [5]$, let $U_{i,s}$ be the indicator of the event that the i-th card is the Ace of suit s. Then $Z_s = \sum_{i=1}^{5} U_{i,s}$. Now $E(U_{i,s}) = P(U_{i,s} = 1)$ which is the probability that the i card is the Ace of suit s. This probability is is $1/52$ because each card has equal probability to be the i-th card. Therefore $E(Z_s) = \sum_{i=1}^{5} E(U_{i,s}) = \sum_{i=1}^{5} 1/52 = 5/52$, as claimed in Eq. (7).