10-23-2021

4.125 $n \geq 1$

O_n odd subsets of $[n]$

E_n even "

$f : O_n \to E_n$

$g : E_n \to O_n$

$f \circ g$: id. of E_n

$g \circ f$: id. of O_n

$\mathrm{switch}_x : \mathcal{P}([n]) \to \mathcal{P}([n])$

$x \in [n]$

$A \subseteq [n]$

$$\mathrm{switch}_x(A) = \begin{cases} A \cup \{x\} & \text{if } x \notin A \\ A \setminus \{x\} & \text{if } x \in A \end{cases}$$

$f = \mathrm{switch}_x|_{O_n}$ $\quad O_n \to E_n$

$g = \mathrm{switch}_x|_{E_n}$ $\quad E_n \to O_n$

4.161 $A, B \subseteq \mathbb{N}_0$ $\quad |A| = 100$

$A \oplus B = \mathbb{N}_0$

$A := \{0, \dots, 99\}$

$B := 100\,\mathbb{N}_0$

Division Thm divisor = 100 rem.

5.03 $|A| = k$

$|B| = \ell$

$$\#\{\text{relations } R \subseteq A \times B\} = 2^{k\ell}$$

$|A \times B| = k\ell$

5.08ab $2^{n/2} < F_n < 2^n$

$n \geq 7$ (b) $n \geq 0$ (a)

0	1	2	3	4	5	6	7
0	1	1	2	3	5	8	13

(a) inductive step: assume $n \geq 2$

$\underline{IH}$ $F_{n-1} < 2^{n-1}$, $F_{n-2} < 2^{n-2}$

$\underline{DC}$ $F_n < 2^n$

$F_n = F_{n-1} + F_{n-2}$

$F_{n-1} < 2^{n-1}$

$F_{n-2} < 2^{n-2}$

$F_n < 2^{n-1} + 2^{n-2} < 2^{n-1} + 2^{n-1} = 2^n$ ✓

$\boxed{n=0 \text{ and } n=1 \text{ go into base case}}$

$n=7$ $2^{7/2} < F_7 = 13$

$128 = 2^7 < 13^2 = 169$ ✓

$8 = 2^{6/2} = F_6 = 8$

Inductive step.

IH $\quad 2^{\frac{n-1}{2}} < F_{n-1}$

$$2^{\frac{n-2}{2}} \leq F_{n-2}$$

$\circledast \quad < 2^{\frac{n-1}{2}} + 2^{\frac{n-2}{2}} \quad < F_n$

$2^{\frac{n}{2}} = 2^{\frac{n-2}{2}} + 2^{\frac{n-2}{2}} < \circledast$ ✓

5.12 $\quad \gcd(F_n, F_{n+1}) = 1$

Base: $n=0$

IH: $\gcd(F_n, F_{n-1}) = 1$

$n \geq 1$

Inductive step:

$$F_{n+1} = F_n + F_{n-1}$$

$$\gcd(F_n, F_{n+1}) = \gcd(F_n, F_n + F_{n-1})$$

$$= \gcd(F_n, F_{n-1}) = 1$$

Euclid's gcd Lemma

I.H.

5.26

$|A| = k \quad |B| = \ell$

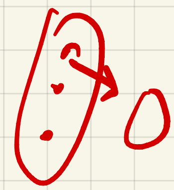
⊛ "$(\forall f \in B^A)(f \text{ is a surjection})$"

f.o. fmla in $(\mathbb{Z}; +, \times, 0, 1, \leq)$

all variables: integers

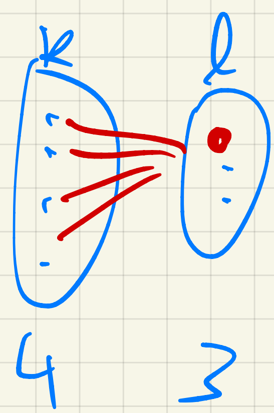
in terms of k, ℓ (free variables)

$(\ell = 1)$ OR $(\ell = 0)\cancel{\wedge}$



$k \geq 1, \ell = 0$ T/F $\quad \emptyset^A = \emptyset$ if $|A| \geq 1$ $\quad \|$ $k = \ell = 0$, $|\emptyset^\emptyset| = 1$ surj. ✓

pred. ex: $\exists$ surj: $\iff (k \geq \ell) \wedge (\ell \geq 1 \vee k = \ell = 0)$



$\forall$ element of $\emptyset$ is a 7-headed dragon

⊛ $\iff \ell \leq 1$

i.e. $\ell = 1 \vee \ell = 0$

5.32 a Smallest $n \in \mathbb{N}_0$

s.t. $(\forall xyz)(n \neq x^2+y^2+z^2)$

$n=7$

$0 = 0^2+0^2+0^2$

$\vdots$

$6 = 2^2+1^2+1^2$

$\forall x, y, z$ $7 \neq x^2+y^2+z^2$

WLOG $x \geq y \geq z \geq 0$

$x \leq 2$ b/c o/w $3^2 > 7$

$y \leq 1$ b/c o/w $2^2+2^2 > 7$

So max can have $2^2+1^2+1^2 = 6 < 7$

$\rightarrow\leftarrow$

5.32 b $\exists \infty$ many n s.t. $n \neq x^2+y^2+z^2$

$n \equiv 7 \mod 8$ OR 28

3.40 f.o. fmla $\varphi = \sigma \vee \tau$

true over $\mathbb{Z}$, $\mathbb{R}$

false over $\mathbb{Q}$

σ: false for $\mathbb{Z}, \mathbb{Q}$, true $\mathbb{R}$

$(\exists x)(x^2=2)$

τ: false for $\mathbb{Q}, \mathbb{R}$, true $\mathbb{Z}$

$(\forall x)(2x \neq 1)$

4.133

surj: $[n] \to [3]$

$$3^n - 3(2^n-2) - 3$$

$$= 3^n - 3\cdot 2^n + 3$$

f: range $[3]$ ✓

range $\{1,2\}, \{1,3\}, \{2,3\}$

$\{1\}, \{2\}, \{3\}$ const: 3

2^n-2

$\boxed{\#\text{surj } [n] \to [2] \quad 2^n-2}$

$A \subseteq \mathbb{Z}, \ |A| = k$

4.100 tight upper bound on $|A+A+A|$

$|A+A+A|$

$A+A+A = \{a+b+c \mid a, b, c \in A\}$

$= b+a+c$

$A = \{a_1 < a_2 < \cdots < a_k\}$

$\{a_i + a_j + a_\ell \mid i \leq j \leq \ell\}$

$\Omega = \{(i,j,\ell) \mid 1 \leq i \leq j \leq \ell \leq k\}$

$f : \Omega \to \mathbb{Z} \qquad f(i,j,\ell) = a_i + a_j + a_\ell$

$\text{range}(f) = A+A+A$

$|A+A+A| \leq |\Omega| = \frac{(k+2)(k+1)k}{6}$ why?

$A \times A \times A$

5.72 Euclid's gcd lemma

$$\mathrm{Div}(a,b) = \mathrm{Div}(a-b,\, b)$$

only thing we need: <u>additivity of divisibility</u>

$$x \mid y \wedge x \mid z \implies x \mid y \pm z$$

$\subseteq$

$d \in \mathrm{Div}(a,b) \Leftrightarrow d \mid a,\ d \mid b \implies d \mid a-b,\ d \mid b \Leftrightarrow d \in \mathrm{Div}(a-b,b)$

$\supseteq$

$e \in \mathrm{Div}(a-b,b) \Leftrightarrow d \mid a-b,\ d \mid b \implies \left.\begin{matrix} d \mid (a-b)+b = a \\ d \mid b \end{matrix}\right\} \Leftrightarrow \mathrm{Div}(a,b)$

5.75 $\quad \mathrm{Div}(a, a+b) \overset{\supseteq}{\underset{\subseteq}{}} \mathrm{Div}(a, a+2b)$

$\subseteq$ true $\quad \| \quad \mathrm{Div}(a,b) \quad$ ✓

$\supseteq$ false

$(\forall a,b)[\mathrm{Div}(a, a+b) \supseteq \mathrm{Div}(a, a+2b)]$ <u>false</u> $\qquad a=0$, $b=1$

$\mathrm{Div}(0,1) \not\supseteq \mathrm{Div}(0,2)$

6.160 FUND THM EQ REL

every ($\forall$) eq. rel. comes from a ($\exists$) partition

$\sim$ is an eq. rel. on Ω

we define the blocks of desired partition

take $a \in \Omega$ $\quad [a] = \{b \in \Omega \mid a \sim b\} \subseteq \Omega$

NTS: If $[a] \cap [b] \neq \emptyset \implies [a] = [b]$ $\quad$ ⊛

$$\bigcup_{a \in \Omega} [a] = \Omega$$

8.48 $\quad a_n \sim b_n \iff a_n - b_n = o(a_n)$

$$\frac{a_n}{b_n} \to 1 \iff \frac{b_n}{a_n} \to 1$$

DEF $c_n = o(a_n)$ if $\frac{c_n}{a_n} \to 0$

i.e. $\quad 1 - \frac{b_n}{a_n} \to 0$

$\Updownarrow$

i.e. $\quad \frac{a_n - b_n}{a_n} \to 0$

$\Updownarrow$

i.e. $\quad a_n - b_n = o(a_n)$

DEF big-Oh $\quad a_n \overset{\text{is}}{=} O(b_n)$ if

$(\exists C)$(for all suff large n)$(|a_n| \le C|b_n|)$

example: $\quad \underbrace{1000n^2 + 10^6 n}_{\le Cn^2 \text{ eventually}} = O(n^2)$

need $\overset{\text{const.}}{C}$ s.t.

any $C > 1000$ works