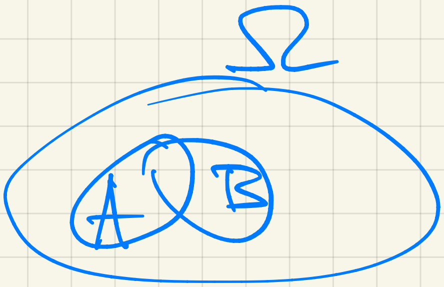
ORD 5b

11-6-2021

9.35



wishlist

$$Pr(A) = \frac{2}{3}$$

$$Pr(B) = \frac{4}{7}$$

$$Pr(A \cap B) = \frac{1}{4}$$

$$Pr(\overline{A \cup B}) = \frac{1}{84} = 1 - P(A) - P(B) + P(A \cap B)$$

min $\leftarrow |\Omega|$

$|\Omega| = 84$ unif

realization

$\Omega = \{1, 2, 3, 4\}$

$A = \{1, 2\}$

$B = \{2, 3\}$

$A \begin{cases} 1 \circ \\ 2 \circ \end{cases}$ $\left.\begin{matrix} 2 \circ \\ 3 \circ \end{matrix}\right\} B$ $4 \circ$

$\frac{2}{3} - \frac{1}{4} =$

$1/4$

$\frac{4}{7} - \frac{1}{4} =$

$\frac{1}{84}$

1

NTS: $|\Omega| \geq 4$

b/c 4 disj. sets

$A \setminus B$
$B \setminus A$
$A \cap B$
$\overline{A \cup B}$

each has ≥ 1 elem. event, they don't share

9.40 # elementary events is a power of 2

$\Pr = 0$ $\qquad$ $\Pr = 1$

- $N := \{a \in \Omega \mid \Pr(a) = 0\}$

for $A \subseteq \Omega$

$\Pr(A) = 0 \iff \boxed{A \subseteq N}$ ← LEMMA

why? $P(A) = \sum_{a \in A} P(a)$

$0 = \sum P(a)$

$P(a) \geq 0$

$(\forall a \in A)(P(a) = 0)$

i.e. $A \subseteq N$

$|N| = k$ •

$\Downarrow$

$\{A \mid \Pr(A) = 0\} = \mathcal{P}(N)$

$|\mathcal{P}(N)| = 2^k$

$\boxed{\Pr(B) = 1 \iff \Pr(\bar{B}) = 0}$

$B \to \bar{B}$ bijection between events of ~~prob~~ 1 and events of $\Pr = 0$

$\downarrow$

such events $= 2^k$

trivial events $= 2^k + 2^k = 2^{k+1}$

9.46 $\Omega = \{0,1\}^n$

$(0,1)$-strings of length n

$|\Omega| = 2^n$

0110111

$s = 1010110$

$= s_1 \cdots s_n$

$k(s) = \sum s_i$

$= \#1s$ in s

$$\boxed{\Pr(s) := p^{k(s)} \cdot (1-p)^{n-k(s)}}$$

DEF

(1) Show: this a prob. distr.

(a) $\Pr(s) \geq 0$ $\quad p \geq 0 \quad (1-p) \geq 0$

$k(s) \geq 0 \quad n-k(s) \geq 0$

✓

(b) $\sum_{s \in \Omega} \Pr(s) = 1$

group together all those strings s with $k(s) = k$

$$\boxed{\Omega_k = \{s \mid k(s) = k\}}$$

So if $s \in \Omega_k$ then $\Pr(s) = p^k (1-p)^{n-k}$ ⊛

$|\Omega_k| = \binom{n}{k}$

Ω_k

$$\sum_{s\in\Omega} \Pr(s) = \sum_{k=0}^{n} \underbrace{\sum_{s\in\Omega_k} \Pr(s)}_{\binom{n}{k}\cdot ⊛} = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = (\underbrace{p + (1-p)}_{1})^n = \underline{\underline{1}}$$

Binomial Thm

✓

$$\Omega = \{0,1\}^n$$

$s = s_1 s_2 \dots s_n$ string $\quad s_i \in \{0,1\}$

$$\Pr(s_i = 1) \underset{\text{Claim}}{=} p$$

NTS

$$\sum_{\substack{s\in\Omega \\ s_i=1}} \Pr(s) = \sum_{\substack{s\in\Omega \\ s_i=1}} p^{k(s)}(1-p)^{n-k(s)}$$

$$\sum_{k=\boxed{0}}^{n} \sum_{\substack{s\in\Omega_k \\ s_i=1}} \underline{p^k(1-p)^{n-k}} = \sum_{k=1}^{n} \text{ same } \otimes$$

how many in Ω_k satisfy $s_i = 1$

$\Omega_k = \{s\in\Omega \mid k(s) = k\}$

\#Heads $= k$

$\Omega'_k = \{s \in \Omega_k \mid s_i = 1\}$

HHTHT$\boxed{\text{H}}$TTTH

$$|\Omega'_k| = \binom{n-1}{k-1}$$

$$\otimes \quad \sum_{k=1}^{n} \binom{n-1}{k-1} p^k (1-p)^{n-k} = \sum_{\ell=0}^{n-1} \binom{n-1}{\ell} p^{\ell+1} (1-p)^{n-1-\ell} =$$

$\ell := k-1$

$$= p\cdot \underbrace{(p+(1-p))^{n-1}}_{1} = \underline{\underline{p}}$$

✓

5.93 3 dice (X_1, X_2, X_3) indep.

$A = \{X_1 = 5\}$

$B = \{X_1 + X_2 + X_3 = 9\}$

$\Omega = [6] \times [6] \times [6] = \{(x_1, x_2, x_3) \mid x_i \in [6]\}$

$|\Omega| = 6^3 = 216$

$P(A) = \frac{1}{6}$

$P(B) = \frac{25}{216}$

$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{3/216}{25/216}$

$\boxed{x_1 \le x_2 \le x_3}$

x_1	x_2	x_3	perms
1	2	6	6
1	3	5	6
1	4	4	3
2	2	5	3
2	3	4	6
3	3	3	1
			25

215 512 521 152

225, 252, 522

$A \cap B$ $5\, x_2\, x_3$ $x_2 + x_3 = 4$

x_2	x_3
1	3
2	2
3	1

9.56 Prob of causes

W = population of patient

A : disease $\underline{A}$ $\qquad A = \{x \in W \mid x \text{ has } \underline{A}\}$

B : " $\underline{B}$

C : some other disease: $\underline{C}$

$R = \{x \in W \mid x \text{ respond to treatment } T\}$

$$P(A \mid R) = \frac{P(A \cap R)}{P(R)} = \frac{P(R \mid A) \cdot P(A)}{P(R \mid A) \cdot P(A) + P(R \mid B) \cdot P(B) + P(R \mid C) \cdot P(C)}$$

$$\boxed{= \frac{42}{47}}$$

$\{A, B, C\}$

$$\Omega = \underline{\underline{W}}$$

10.60 $\quad X$ r.v. $\quad X : \Omega \to \mathbb{R}$

(a) $m := \min X \le E(X) \le \max X := M$

$$E(X) = \sum_{a \in \Omega} X(a) \cdot P(a) \ge$$

$$\ge \sum_{a \in \Omega} m \cdot P(a) = m \underbrace{\sum_{a \in \Omega} P(a)}_{1} = m \quad ✓$$

$X := -X$ reduces max question to min question ✓

(b) is it possible that $m = E(X) < M$?

YES

Ω	Pr	X
a	1	$m := 0$
b	0	$M := 1$

$E(X) = m = 0$
$< M = 1$

10.68 mismatched envelopes

n letters $\longrightarrow$ n $\boxtimes$

X: # letters in correct $\boxtimes$

$E(X)$

$$Y_i = \begin{cases} 1 & \text{if } \boxed{i^{th}} \text{ letter is correctly matched} \\ 0 & \text{o/w} \end{cases}$$

$$X = \sum_{i=1}^{n} Y_i$$

$$E(X) = \sum E(Y_i) = \sum P(Y_i = 1)$$

$$P(i^{th} \text{ letter matched}) = \frac{1}{n}$$ for reasons of symmetry

$$\therefore E(X) = \sum_{i=1}^{n} \frac{1}{n} = n \cdot \frac{1}{n} = \underline{\underline{1}}$$

Sample space $\boxed{n!}$

= all bijections {letters} $\longrightarrow$ { $\boxtimes$s }

6.175 $B(n)$ = # partitions of $[n]$

(a) $B(n) \le n^n$ surjective

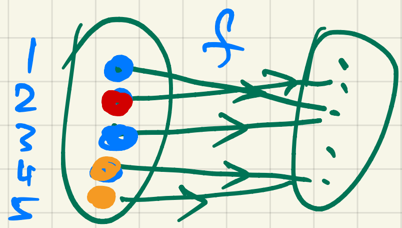
(b) $B(n) \le n!$ injective

$F: [n]^{[n]} \xrightarrow{\text{onto}} \{\text{partitions of } [n]\}$

pick $f \in [n]^{[n]}$ $\quad f : [n] \to [n]$

define partition $F(f)$ of $[n]$

$\boxed{F := \ker}$

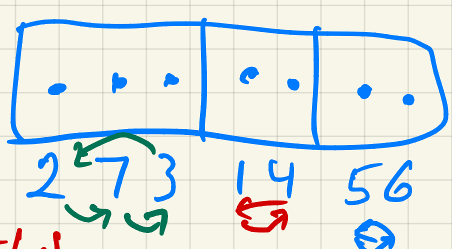


$\pi = \{\{1 3\}, \{2\}, \{4 5\}\} = \ker(f)$

kernel of f

(a) ✓

(b)



$[7]$

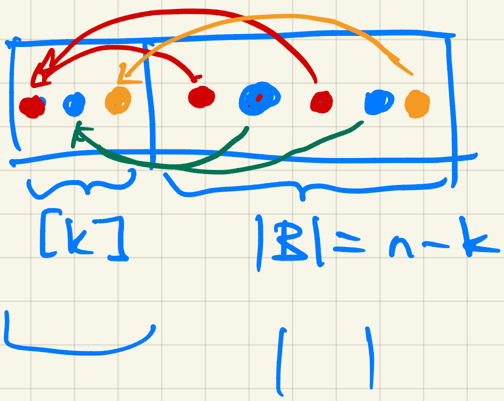
x	$\pi(x)$
1	4
2	7
3	2
4	1
5	6
6	5
7	3

$(\forall k)(1 \le k \le n)$

(c) $B(n) \ge k^{n-k}$ inj:

$f : [n-k] \to [k]$

find $G(f)$: partition $f \ne g \Rightarrow G(f) \ne G(g)$



$f : B \to [k]$

8.25

8.25 $\quad p_n \sim n \ln n$

p_n : nth prime

use PNT $\quad \pi(x) \sim \dfrac{x}{\ln x}$

$\pi(x) = \#$ primes $\leq x$

$(*)$ $\dfrac{p_n}{\ln p_n} \sim \boxed{\pi(p_n) = n} \to \infty$

$$\ln \frac{p_n}{\ln p_n} \sim \ln n$$

$$\ln p_n - \ln \ln p_n \sim \ln n$$

$$1 - \underbrace{\frac{\ln(\ln p_n)}{\ln p_n}}_{\to 0} \sim \frac{\ln n}{\ln p_n}$$

$$\frac{\ln n}{\ln p_n} \to 1$$

p_1	p_2	p_3	p_4	p_5
2	3	5	7	11

$\pi(11) = 5$

$\pi(p_5) = 5$

$$\lim_{x \to \infty} \frac{\ln x}{x} = 0$$

$x := \ln p_n \to \infty$

i.e. $\quad \ln n \sim \ln p_n \quad \leftarrow$ LEMMA

$(*)$ $$n \sim \frac{p_n}{\ln p_n} \sim \frac{p_n}{\ln n}$$

$$n \ln n \sim p_n$$