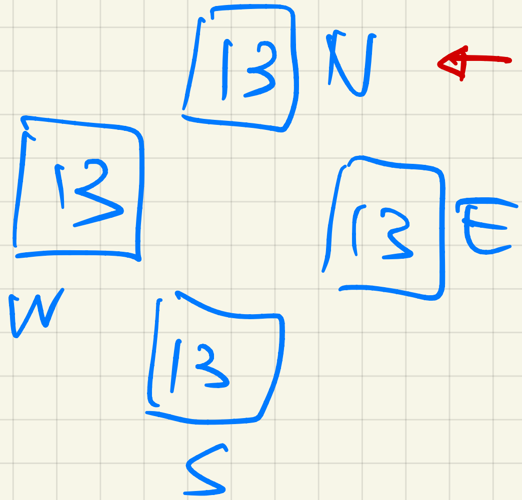
9.49 bridge

11-13-2021

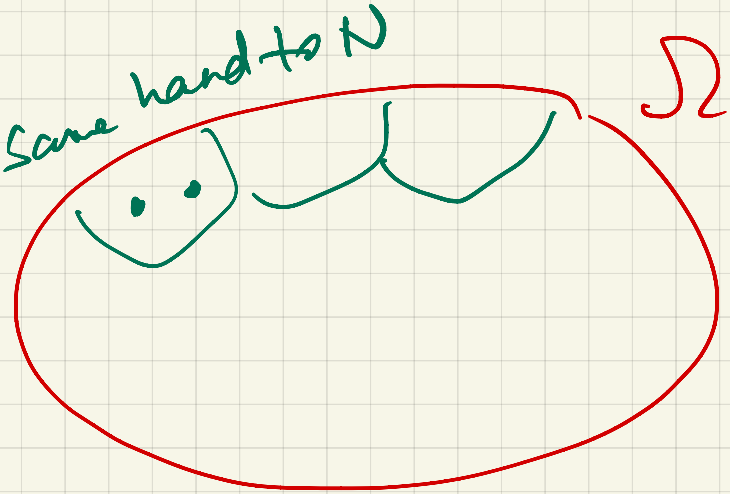
(a) all Aces go to N

(b) each player gets an Ace



(a) $$\frac{\binom{48}{9}\cdot\binom{39}{13}\binom{26}{13}}{\binom{52}{13}\cdot\binom{39}{13}\binom{26}{13}}$$

$$|\Omega| = \binom{52}{13}\overbrace{\binom{39}{13}\binom{26}{13}}^{\text{each block}}\binom{13}{13}$$



(b)

$$= \frac{13^4\cdot 4!}{\binom{52}{4}\cdot 4!}$$

$$\underbrace{\square\,\boxed{A}\,\square}_{13}\ \underbrace{A\ \cdots}_{13}\ \underbrace{A}_{13}\ \underbrace{\boxed{A}}_{13}$$

$|\Omega| = 52!$

(b)

$$\frac{4! \cdot \binom{48}{12}\binom{36}{12}\binom{24}{12}\binom{12}{12}}{\binom{52}{13}\cdot\binom{39}{13}\cdot\binom{26}{13}\cdot\binom{13}{13}}$$

$$= \frac{13^4}{\binom{52}{4}}$$

10.33

$$A \cap B \subseteq A \cup B$$

Let's make $A \cup B = \Omega$

$\boxed{\Omega = \{1, 2\} \quad \text{Pr uniform}}$ $\boxed{A = \{1\} \quad B = \{2\}}$

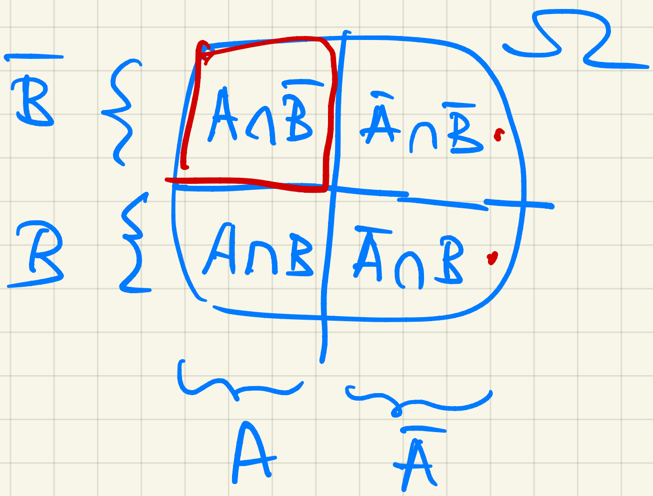
A, B nontriv. indep events $\implies |\Omega| \geq 4$

Lemma

$\Downarrow$

$A, \bar{B}$ indep

$\therefore \bar{A}, \bar{B}$,



$$\Omega = (A \cap B) \,\dot{\cup} \ldots \dot{\cup}\, (\bar{A} \cap \bar{B})$$

$$\bar{A} \cap B \subseteq B$$

$$\bar{A} \cap \bar{B} \subseteq \bar{B}$$

$$\therefore (\bar{A} \cap B) \cap (\bar{A} \cap \bar{B}) = \emptyset$$

$$(\bar{A} \cap \bar{A}) \cap \underbrace{(B \cap \bar{B})}_{\emptyset}$$

Claim: each of these 4 subsets is non-empty

e.g. $A \cap \bar{B} \neq \emptyset$

b/c $A, \bar{B}$ indep

$$\implies \Pr(A \cap \bar{B}) = \underbrace{\Pr(A)}_{\neq 0} \cdot \underbrace{\Pr(\bar{B})}_{\neq 0} \neq 0$$

10.75 Club of 2000

$$X_i = \begin{cases} 1 & \text{if } \underline{\underline{\text{member}}} \# \underline{\underline{i}} \text{ is lucky} \\ 0 & \text{o/w} \end{cases}$$

$$|\Omega| = 2000!$$

$$P(X_i = 1) = P(i^{th} \text{ member is lucky})$$

$$= \frac{1999!}{2000!} = \frac{1}{2000}$$ — b/c vodka

$Y = \#$ lucky members

$$Y = \sum X_i$$

$$E(Y) = \sum E(X_i) = \sum P(X_i = 1) = 2000 \cdot \frac{1}{2000} = \underline{\underline{1}}$$

$$Z_i = \begin{cases} 1 & \text{cards } \# i \text{ goes to lucky member} \\ 0 & \text{o/w} \end{cases}$$

$$P(Z_i = 1) = \frac{c_i}{2000}$$ $\Leftarrow \#$ members

$c_i = \#$ people born in year i

$$Y = \sum Z_i$$

$$E(Y) = \sum E(Z_i) = \frac{\sum c_i}{2000} = \underline{\underline{1}}$$ ✓

$$\sum_{i=1}^{2000} c_i = 2000$$ b/c vodka

12.35 for which graphs is "adjacent or equal" an equiv. rel

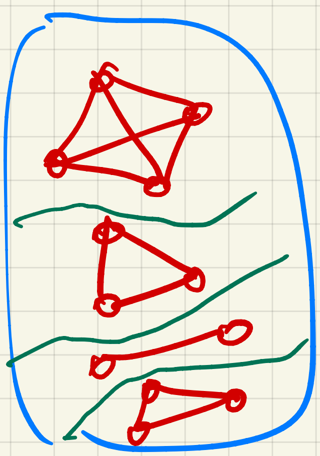
always reflexive b/c "or equal"

symm.

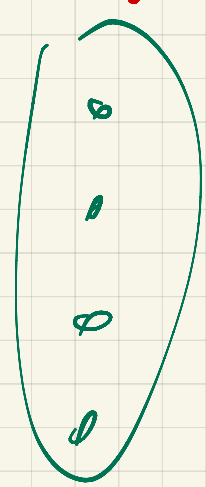
when is it transitive?

if G complete $\Rightarrow$ ✓

?



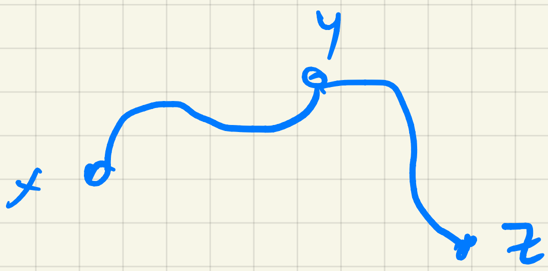
$\overline{K_n}$

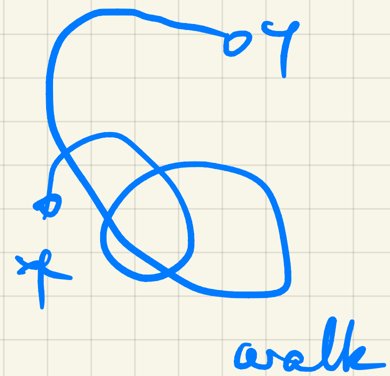


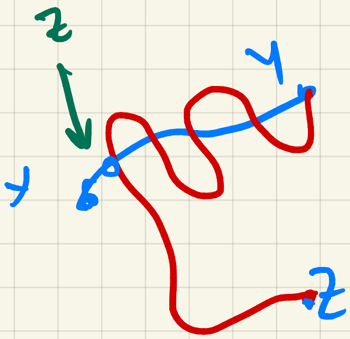
disjoint union of complete graphs

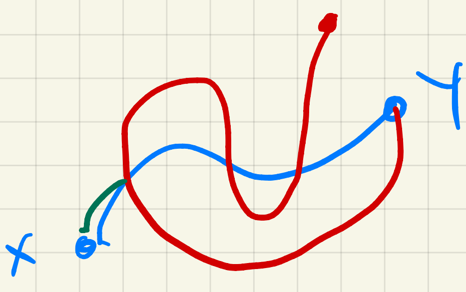
graphs s.t. every connected comp. is a clique

12.38 path-accessibility is an eq. rel.

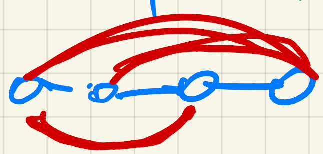


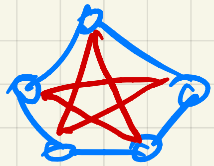


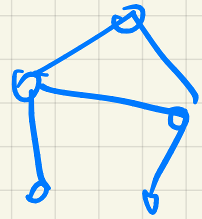




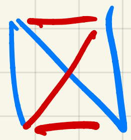
11.70 $\qquad G \cong \overline{G}$

P_4 

C_5 

So 

self-compl.



Claim $\quad n \equiv 0$ or $1 \bmod 4$

$$m_G + m_{\overline{G}} = \binom{n}{2}$$

$$\therefore \quad 2m_G = \binom{n}{2}$$

$$2 \,\Big|\, \frac{n(n-1)}{2} \qquad \text{i.e.} \quad 4 \mid n(n-1)$$

① $4 \mid n$ $\iff n \equiv 0 \ (4)$

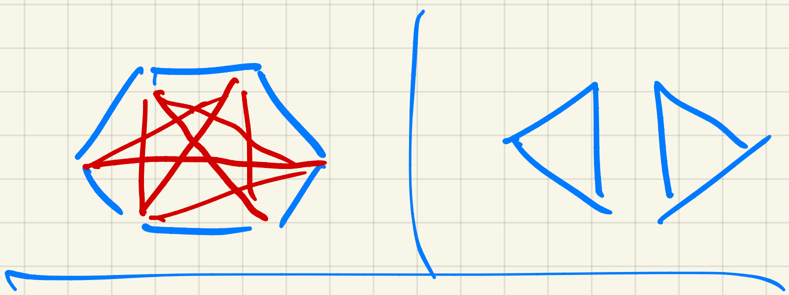
② $4 \mid n-1$ $\iff n \equiv 1 \ (4)$

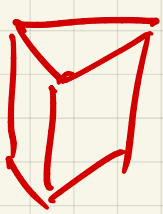
③ $2 \mid n$

$2 \mid n-1$

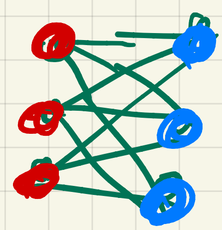
$2 \mid n-(n-1) = 1 \rightarrow\leftarrow$

11.50 non-iso graphs, regular of same deg, same # vert., smallest





$\not\cong$ $K_{3,3}$



$2K_3$ $P \cdot K_3$ bipartite

12.40 # paths of length k in K_n

P_{k+1}

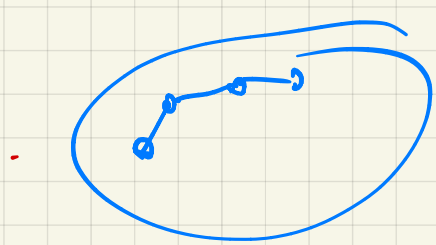
$$\binom{n}{k+1}\cdot\frac{(k+1)!}{2}$$

$v = k+1$

> have v vertices

ways to make it a path?

$v\cdot(v-1)\cdots1 = v!$



overcounted twice

$$\frac{1}{2}\underbrace{n(n-1)\cdots(n-k)} = \frac{1}{2}\cdot\binom{n}{k+1}(k+1)!$$

10.80 $\overset{n}{\text{marbles}}$ in $\overset{n}{\text{cups}}$

$$X_i = \begin{cases} 1 & \text{if cup } \# i \text{ ends up empty} \\ 0 & \text{o/w} \end{cases}$$

$Y = \sum X_i$ # empty cups

$$E(X_i) = \Pr(i^{th} \text{ cup empty}) = \left(1 - \frac{1}{n}\right)^n$$

$$E(Y) = n \cdot \left(1 - \frac{1}{n}\right)^n$$

$$\left(1 - \frac{1}{n}\right)^n \longrightarrow \frac{1}{e}$$

$\forall x \in \mathbb{R}$ $\displaystyle \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = e^x$

$$\boxed{E(Y) \sim \frac{n}{e}}$$

$\Omega = \{\text{cups}\}^{\{\text{marbles}\}}$

$f : \{\text{marbles}\} \to \{\text{cups}\}$

$|\Omega| = n^n$

6.1.12

$$b_n = 2^{\binom{n}{2}} \qquad a_n = \frac{b_n}{n!}$$

<u>Claim</u> $\ln b_n \overset{?}{\sim} \ln a_n$

$$\underbrace{\binom{n}{2}\cdot \ln 2}_{x_n} \overset{?}{\sim} \underbrace{\binom{n}{2}\cdot \ln 2}_{x_n} - \underbrace{\ln(n!)}_{z_n}$$

$$n! < n^n$$

$$\ln(n!) < n \ln n$$

$$\underline{x_n \sim x_n - z_n}$$

$$\underline{x_n} > x_n - z_n > \underline{x_n - n \ln n}$$

$$1 > 1 - \frac{z_n}{x_n} > 1 - \underbrace{\frac{n \ln n}{x_n}}_{\to 0}$$

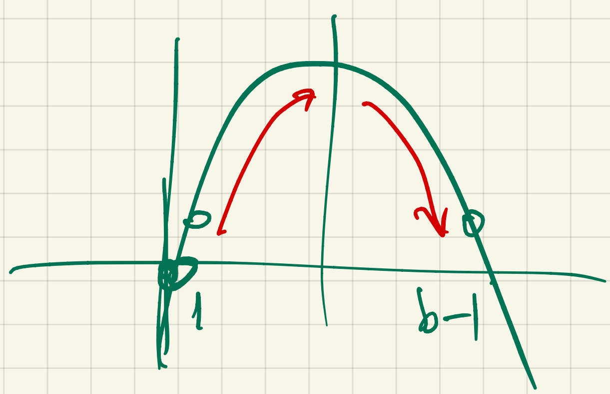
6.1.15 $\quad Q_d \quad$ d-dim cube

$n = 2^d$

Given $\quad x+y = b > 0$

$$\frac{\max(x \cdot y)}{\min(x \cdot y) = \min\{x(b-x) \mid 1 \le x \le b-1\}}$$

$x(b-x) \ge b-1$



12.18

Ω	Pr	X	Y
1	1/3	-1	1
2	1/3	0	0
3	1/3	1	1

$E(X) = 0$ $\qquad$ $Y := X^2$

$XY = X$

$E(XY) = E(X) = 0$

$$\left.\begin{array}{l} E(XY) = \underline{\underline{0}} \\ E(X)\cdot E(Y) = 0\cdot E(Y) = \underline{\underline{0}} \end{array}\right] \text{uncorrelated}$$

not indep.

$P(X=1) = \frac{1}{3}$

$P(Y=0) = \frac{1}{3}$

$P(X=1 \wedge Y=0) = 0 \neq \frac{1}{3}\cdot\frac{1}{3}$

$\therefore$ X, Y not indep

10.70

$E(\#\text{Aces in poker hand})$

$|\Omega| = \binom{52}{5}$

$X_i = \begin{cases} 1 & i^{th} \text{ card Ace} \\ 0 & \text{o/w} \end{cases}$

$Y = \sum_{i=1}^{5} X_i$

$E(X_i) = P(i^{th} \text{ card Ace}) = \frac{4}{52} = \frac{1}{13}$

$E(Y) = 5 \cdot \frac{1}{13} = \frac{5}{13}$

Wrong sample space for this argument

$|\Omega'| = 52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 = \binom{52}{5} \cdot 5!$

blowing up sample space

$Z_j \; : \; j \in \{\spadesuit, \heartsuit, \diamondsuit, \clubsuit\}$ four suites

indic. ace of suit j is in poker hand

$P(Z_j = 1) = \frac{5}{52}$

$E(Y) = 4 \cdot \frac{5}{52}$