11.60 If G k-regular $\overset{?}{\Longrightarrow}$ $n \mid m$

11-20-2021

Handshake

$nk = 2m$

$\therefore 2 \mid nk$

$\Rightarrow 2 \mid n$ or $2 \mid k$

(a) n odd

$\Rightarrow 2 \mid k$

$nk = 2m$

$n \cdot \frac{k}{2} = m \qquad \therefore n \mid m$

(b) n even

$n = 2 \nmid m = 1$

K_2 is a counterexample

11.65 $n \geq 2 \Longrightarrow \exists$ 2 vertices of same degree

$0 \leq \deg \leq n-1$

all deg $\subseteq \{\overbrace{0, \ldots, n-2}^{n-1}\}$

or $\subseteq \{1, \ldots, n-1\}$

If $(\exists x)(\deg x = 0)$

then $(\nexists y)(\deg y = n-1)$

$\left.\begin{array}{l} |\text{range } \deg| \leq n-1 \\ |\text{domain}(\deg)| = n \end{array}\right\} \Rightarrow$ collision

PHP

14.20 2 copies of Petersen's graph

(b) name of 2nd : - " -

14.27 [If G is k-regular, girth ≥ 5 $\Rightarrow n \geq k^2+1$] BONUS

$n = k^2+1$ possible? YES for $k=1,2,3$

K_2, C_5, Petersen's

THM: If $n=k^2+1$ then $n \in \{1,2,3,7,\underline{57}\}$

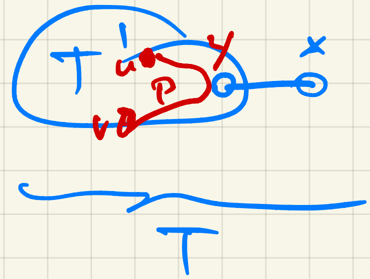
14.32 $\forall$ tree $m=n-1$

Induction on n

Inductive step: we are given $n \geq 2$

Inductive hyp: then true for all $n' < n$

Lemma. tree w ≥ 2 vert. has a vertex of deg 1



$\deg x = 1$

$T \setminus x = T'$

$n' = n-1$

$m' = m-1$

by I.H. $m' = n'-1$

$m-1 = (n-1)-1$

$m = n-1$

NTS: $x \notin P$

x could only be end pt of P (b/c $\deg = 1$) but $x \notin V(T')$

I.H. tree T' w. $n' < n$ vertices $\Rightarrow m' = n'-1$

T' has no cycles ✓

T' is conn.

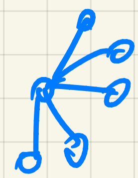
14.36 G conn. $\implies m \geq n-1$

Pf: $\exists$ spanning tree T $\quad |V(T)| = n$

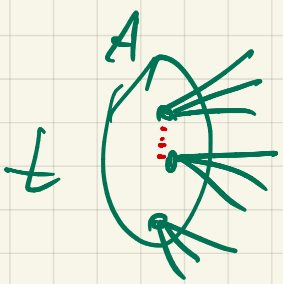
$|E(T)| = n-1 \qquad E(T) \subseteq E(G)$ ✓

14.75 $\exists$ connected graph $\alpha(G) = n-1$

tight: $\alpha(G) = n \iff G = \overline{K_n}$

example:  $St_n = K_{1,n-1}$

14.78 If G regular of deg $= k \geq 1$
$\implies \alpha(G) \leq n/2$



$|A| = t =$ size of an indep. set

kt edges leave t

$\Rightarrow m \geq k \cdot t$

$m = \frac{kn}{2}$

$\frac{kn}{2} \geq kt \qquad \div k \quad > 0$

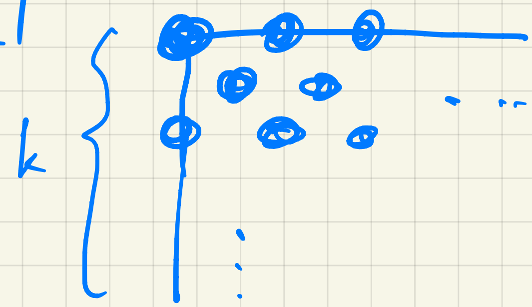
$\frac{n}{2} \geq t$

~~$\alpha \leq \frac{n}{k+1}$~~ 2

$\alpha \geq \frac{n}{k+1}$

$k = deg_{max}$

14.93 $\alpha(\mathrm{grid}(k,\ell)) = \left\lceil \frac{k\ell}{2} \right\rceil$



lower bound

Case If k even:

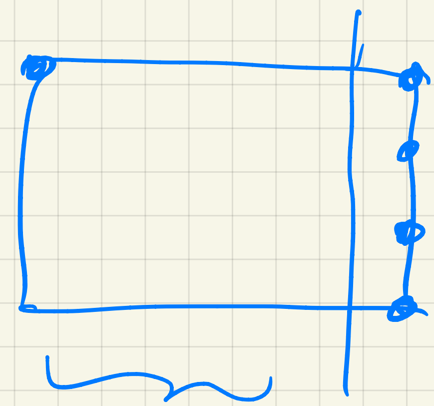
$\frac{k}{2}$ vertices in each column

$\Rightarrow \frac{k}{2} \cdot \ell = \frac{k\ell}{2} = \left\lceil \frac{k\ell}{2} \right\rceil$ ✓ found an indep set of size $\left\lceil \frac{k\ell}{2} \right\rceil$

Case same of ℓ even

Case $k\ell$ odd

$k(\ell-1)$ even



$k \times (\ell-1)$

$\frac{k(\ell-1)}{2} + \frac{k+1}{2} =$

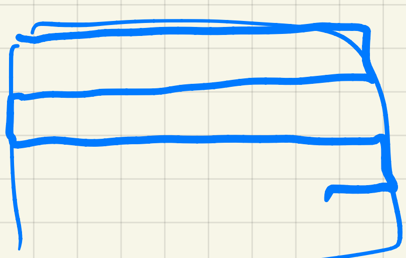
$= \underbrace{\frac{k\ell}{2}}_{\text{odd}/2} + \frac{1}{2} = \left\lceil \frac{k\ell}{2} \right\rceil$

upper bound

Lemma $\mathrm{Grid}(k,\ell) \supseteq P_{k\ell}$ – spanning

Hamilton path

Snake-like



Lemma If H is a spanning subgraph of G then $\alpha(H) \geq \alpha(G)$

$\therefore \alpha(\mathrm{Grid}(k,\ell)) \geq \alpha(P_{k\ell}) = \left\lceil \frac{k\ell}{2} \right\rceil$

14.140 $\alpha(G)\cdot\chi(G) \geq n$

independence number — chromatic number

Pf. Take an optimal coloring w/ $r=\chi(G)$ colors

$C_i = \{\text{vertices of color } i\}$ indep.

$\therefore |C_i| \leq \alpha(G)$

$$V = C_1 \dot\cup \dots \dot\cup C_r$$

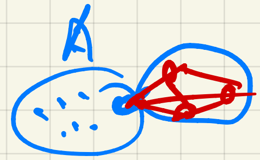
$$n = |V| = \sum |C_i| \leq r\cdot\alpha = \alpha\cdot\chi$$

14.142 $(\forall n)(\exists G)\left(\alpha\cdot\chi \geq \frac{n^2}{4}\right)$

A : $\lceil \frac{n}{2} \rceil$ vert.

B : $\lceil \frac{n}{2} \rceil$ vert.

$|A \cap B| \leq 1$



make A indep.

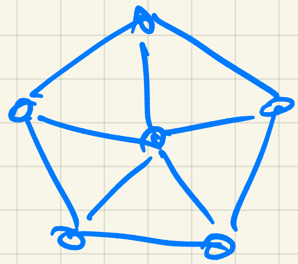
make B clique

$$\alpha \geq \left\lceil \frac{n}{2} \right\rceil$$

$$\chi \geq \left\lceil \frac{n}{2} \right\rceil$$

$$\alpha\cdot\chi \geq \left\lceil \frac{n}{2} \right\rceil \cdot \left\lceil \frac{n}{2} \right\rceil \geq \frac{n}{2}\cdot\frac{n}{2} = \frac{n^2}{4}$$

14.150 find smallest 4-chrom. graph $\not\supset K_4$



NTS not 3-colorable

Pf by contradiction

Suppose $f: V \to \{1, 2, 3\}$ is a legal 3-coloring

$f|_{C_5}$ uses up ≥ 3 colors b/c $\chi(C_5) = 3$

$\Longrightarrow$ no color is legal for center

$\rightarrow\leftarrow$

15.20 $G(n,p)$

edges $= X$

$$X = \sum_{i=1}^{\binom{n}{2}} Y_i$$

$E(X)$

$= \sum E(Y_i)$

$= \binom{n}{2} \cdot p$

the Y_i are indep

$$\Rightarrow \operatorname{Var} X = \sum_i \operatorname{Var}(Y_i) + \sum_{i \neq j} \operatorname{Cov}(Y_i, Y_j)$$

Suffices if $(\forall i \neq j)(\operatorname{Cov}(Y_i, Y_j) = 0)$

i.e. Y_i, Y_j are uncorrelated

Suffices: the Y_i are pairwise independent

$$\operatorname{Var}(Y_i) = E(Y_i^2) - (E(Y_i))^2 = p - p^2 = p(1-p)$$

$Y_i^2 = Y_i$

$$\operatorname{Var} X = \binom{n}{2} p(1-p)$$

15.23

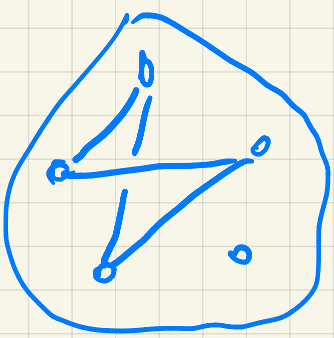
$E(\#\,k\text{-cycles}) = N(n,k)\cdot p^k$

$E(\#\,\text{induced } k\text{-cycles}) = N(n,k)\cdot p^k\cdot$

$\#\,k$-cycles in K_n is $\dfrac{n(n-1)\cdots(n-k+1)}{2k} = N(n,k)$

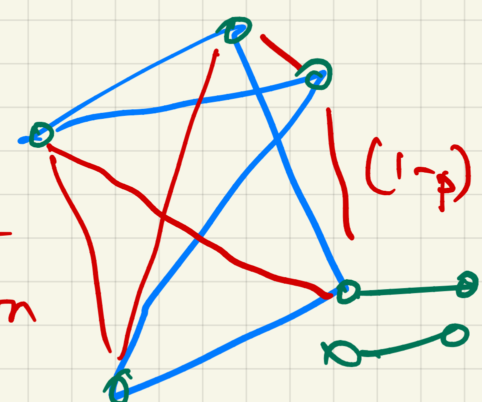
$P($a given k-cycle in K_n appear in $G) = p^k\cdot(1-p)$

random graph



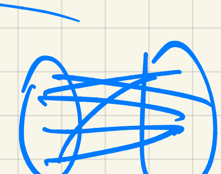
complement w.r.to its own vertices

$|E(\overline{C_k})| = \binom{k}{2} - k$



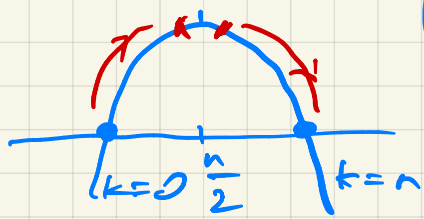
15.20 max # edges of bipartite graph

$\lfloor \frac{n^2}{4} \rfloor$



max $k\cdot(n-k)$

$1 \le k \le n-1$



n even: $\frac{n}{2}\left(n-\frac{n}{2}\right) = \frac{n^2}{4}$

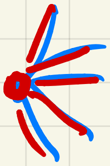
n=odd $\frac{n-1}{2}\cdot\frac{n+1}{2} = \frac{n^2-1}{4} = \lfloor \frac{n^2}{4} \rfloor$

12.50 max # edges in disconnected graph G

If G disc. $\Longrightarrow$ $\bar{G}$ conn $\Longrightarrow$ $|E(\bar{G})| \geq n-1$

$$\Longrightarrow |E(G)| \leq \binom{n}{2} - n - 1 = \binom{n-1}{2}$$

$$G = \overline{S_n}$$



K_{n-1}

$$(\underbrace{A \cap \bar{B} \cap \bar{C}}_{\subseteq \bar{C}}) \cap (\underbrace{\bar{A} \cap \bar{B} \cap C}_{\subseteq C}) \subseteq \bar{C} \cap C = \emptyset$$

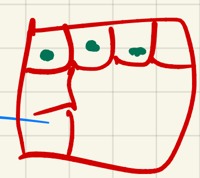
11.17 $A_1 \dots A_k$ nontriv. indep events

$$\Longrightarrow |\Omega| \geq 2^k$$

Pf Lemma $(\forall \varepsilon_1 \dots \varepsilon_n \in \{\pm 1\})(A_1^{\varepsilon_1}, \dots, A_k^{\varepsilon_k}$ indep.$)$

where $A^{+1} = A$, $A^{-1} = \bar{A}$

2^k events $A_1^{\varepsilon_1} \cap \dots \cap A_k^{\varepsilon_k}$



2^k

Claim 1: these events are nontrivial   b/c inters. of nontriv. events

Claim 2: —"— —"— disjoint (pairwise)