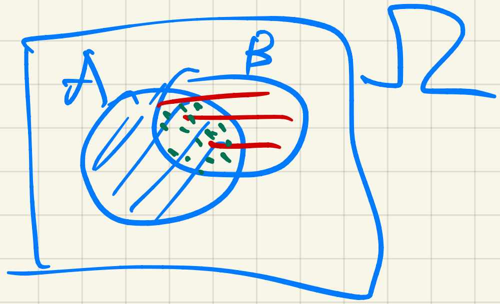
11-30-2021

A, B know: $Pr(A), Pr(B), Pr(A\cap B)$

Q: $Pr(A\cup B) =$



$= \underline{Pr(A)} + \underline{Pr(B)} - \underline{Pr(A\cap B)}$

$A, B, C \subseteq \Omega$

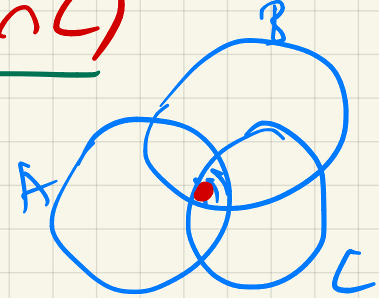
given $\begin{cases} Pr(A), Pr(B), Pr(C) \\ Pr(A\cap B), Pr(A\cap C), Pr(B\cap C) \\ Pr(A\cap B\cap C) \end{cases}$

Q: $Pr(A\cup B\cup C) = ?$

S_1: $\underline{Pr(A) + Pr(B) + Pr(C)}$

$-S_2$: $-\underline{Pr(A\cap B)} - \underline{Pr(A\cap C)} - \underline{Pr(B\cap C)}$

S_3: $+Pr(\underline{A\cap B\cap C})$



$P(\overline{A\cup B\cup C}) = \boxed{1} - S_1 + S_2 + S_3$

$\underbrace{1}_{S_0} = Pr(\Omega)$

2^3 terms

INCLUSION-EXCLUSION formula

$A_1, \dots, A_k \subseteq \Omega$

$I \subseteq [k]$

$$\boxed{A_I = \bigcap_{i \in I} A_i} \qquad \overset{\text{DEF}}{\boxed{A_\emptyset = \Omega}}$$

data: $\forall I \subseteq \Omega$

we are given $\Pr(A_I)$

Q $B := \bigcup_{i=1}^{k} A_i$ complement

$$P(B) = ?$$

$$S_1 = \sum \Pr(A_i)$$

$$S_2 = \sum_{i<j} \Pr(A_i \cap A_j)$$

$$S_3 = \sum_{i<j<l} \Pr(A_i \cap A_j \cap A_l)$$

$$S_i = \sum_{\substack{I \subseteq [k] \\ |I| = i}} \Pr(A_I)$$

i-wise intersections

THM $P(B) = S_0 - S_1 + S_2 - \cdots + (-1)^k S_k$

$S_0 = 1$

$$B = \overline{\bigcup_{i=1}^{k} A_i}$$

$I \subseteq [k]$

$$A_I = \bigcap_{i \in I} A_i$$

$$\Pr(B) = \sum_{I \subseteq [k]} \pm \Pr(A_I)$$

$$(-1)^{|I|}$$

THM (Inclusion-Exclusion rephrased)

$$\boxed{\Pr(B) = \sum_{I \subseteq [k]} (-1)^{|I|} \cdot \Pr(A_I)}$$

terms: 2^k

$$\boxed{\prod_{i=1}^{k} (1+x_i) = \sum_{I \subseteq [k]} \prod_{i \in I} x_i}$$

$$\underbrace{(1+x_1)}\underbrace{(1+x_2)}\underbrace{(1+x_3)} = 1 + x_1 + x_2 + x_3 + x_1 x_2 + x_1 x_3 + x_2 x_2 + x_1 x_2 x_3$$

$x_1 x_3$

$$(1-x_1)\cdots(1-x_3) = 1 - x_1 - x_2 - x_3 + x_1 x_2 + x_1 x_3 + x_2 x_3 - x_1 x_2 x_3$$

$$(*) \quad \prod_{i=1}^{k} (1-x_i) = \sum_{I \subseteq [k]} \pm \prod_{i \in I} x_i$$

$(-1)^{|I|}$

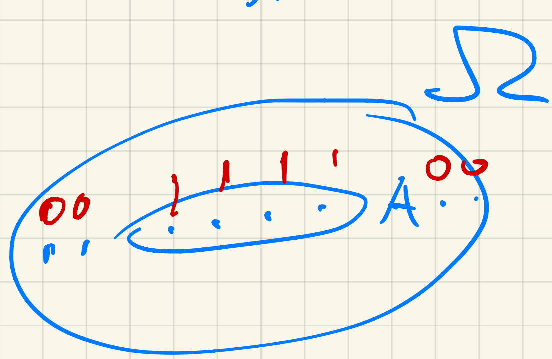
indicator variable: $(0,1)$-valued

$A, B \subseteq \Omega$

$$\vartheta_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{o/w} \end{cases} \qquad (x \in \Omega) \qquad \vartheta_A : \Omega \to \{0,1\}$$

$$\vartheta_A \cdot \vartheta_B = \vartheta_{A \cap B}$$

$$\prod_{i \in I} \vartheta_{A_i} = \vartheta_{\bigcap_{i \in I} A_i}$$



$$B = \overline{\bigcup_{i=1}^{k} A_i} = \bigcap_{i=1}^{k} \overline{A_i}$$

DeMORGAN's law

$$\vartheta_B = \prod_{i=1}^{k} \vartheta_{\overline{A_i}} = \prod_{i=1}^{k} \left(1 - \vartheta_{A_i}\right) =$$

$$\vartheta_B = \sum_{I \subseteq [k]} (-1)^{|I|} \cdot \underbrace{\prod_{i \in I} \vartheta_{A_i}}_{\vartheta_{\underbrace{\bigcap_{i \in I} A_i}_{A_I}}} = \sum_{I \subseteq [k]} (-1)^{|I|} \cdot \underbrace{\vartheta_{A_I}}_{\text{r.v.}}$$

E

$$\underline{\Pr(B)} = E(\vartheta_B) = \sum_{I \subseteq [k]} (-1)^{|I|} \cdot \underbrace{E\left(\vartheta_{A_I}\right)}_{\Pr(A_I)}$$

$\boxed{\text{DO}}$ alphabet $\Sigma = \{A, B, \dots\}$

$|\Sigma| = k$

$W_n = \{$words of length n that involve all the k letters$\}$

example: $k=3$ $\Sigma = \{A, B, C\}$ $n=5$

ACBCC ✓

ACACC ✗

Q $|W_n| = ?$ function of n and k

for every fixed: closed-form expression

(# terms in sum depends on k only)

simple

Incl-Excl.

$$\Pr(\overline{B}) = S_0 - S_1 + S_2 - + \dots + (-1)^k S_k$$

$$S_i = \underbrace{\sum_{\substack{I \subseteq [k] \\ |I| = i}} \Pr(A_I)}_{\#\text{terms}: \binom{k}{i}}$$

$I \subseteq [k]$

$A_I = \bigcap_{i \in I} A_i$

DO*

$\Pr(\overline{B}) \le S_0 \quad = 1$

$\Pr(\overline{B}) \ge S_0 - S_1$

$\Pr(\overline{B}) \le S_0 - S_1 + S_2$

$\Pr(\overline{B}) \ge S_0 - S_1 + S_2 - S_3$

$\vdots$

BONFERRONI'S INEQUALITIES

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estimating $\Pr(B)$ knowing $S_0, \dots, S_t$

LINIAL-NISAN: If $t = o(\sqrt{k})$

then no estim. possible:

If $t = \omega(\sqrt{k})$ then $\forall \varepsilon$ we can estimate

$\Pr(B)$ within $1 \pm \varepsilon$

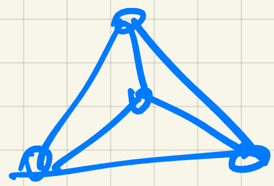
Chebyshev polynomials

Planar graphs

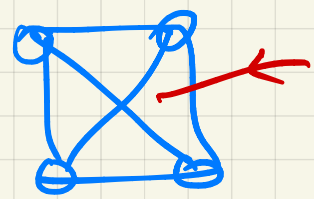
(1) Plane graph: drawing w/o self-intersections

plane realization of a given graph

K_4 :



YES

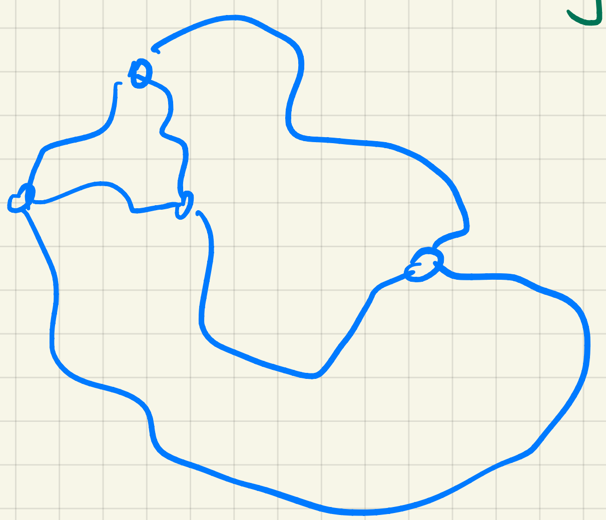


NO

edges: continuous curves

JORDAN CURVES

K_4

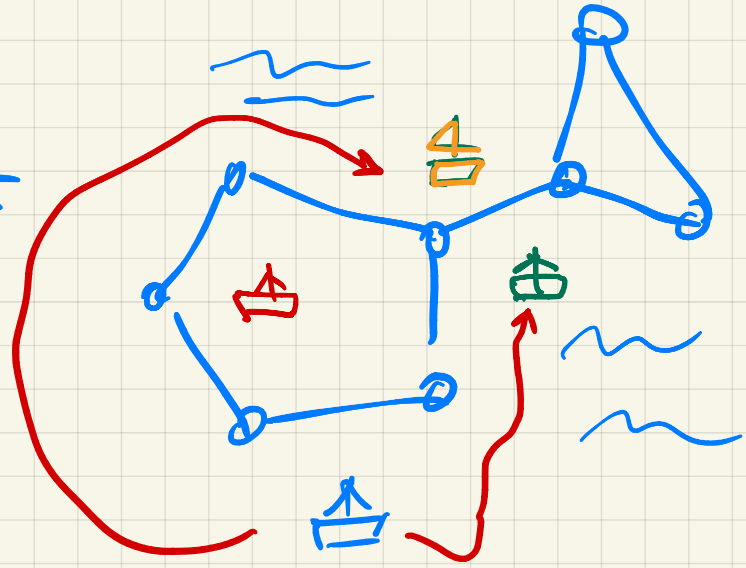


(2) DEF G is planar if $\exists$ plane realization

K_4 is a planar graph

THM K_5, $K_{3,3}$ not planar

Plane graph G



edges : dams

$\mathbb{R}^2 \setminus G$

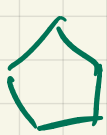
connected components : regions
faces

THM $\boxed{\text{Euler's fmla}}$

If G is a plane realization of a connected graph

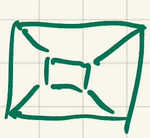
then $\boxed{n - m + r = 2}$

#vertices #edges #regions



$n - n + 2 = 2$

$n - (n-1) + 1 = 2$

cube 

$8 - 12 + 6 = 2$

THM (follows from Euler's fmla)

If G is a planar graph, $n \geq 3$

$\Rightarrow$ $\boxed{m \leq 3n-6}$

if G is a bipartite planar graph, $n \geq 3$

$\Rightarrow$ $\boxed{m \leq 2n-4}$

COROLLARY K_5, $K_{3,3}$ not planar

$\boxed{\text{DO}}$

$\boxed{\text{DO}}$ Every planar graph has a vertex of degree ≤ 5

$\boxed{\text{DO}}$ Every planar graph is 6-colorable

(do not use the 4-color theorem: $\forall$ planar graph is 4-colorable)