12-02-2021

§4.45 girth of toroidal grid $C_k \square C_\ell \leq 4$

if girth 3 then $k=3$ or $\ell=3$ ⊗

$\Leftarrow$ ✓ LOGIC

§4.45 conn. graph G sp. tree T

$\operatorname{diam} G \leq \operatorname{diam} T$

LEMMA. $\rho_G(x,y) \leq \rho_T(x,y)$

$$\underbrace{\max_{x,y \in V} ''}_{\operatorname{diam} G} \leq \underbrace{\max_{x,y \in V} ''}_{\operatorname{diam} T}$$

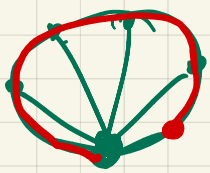
BONUS If G conn then $\exists$ sp. tree T

s.t. $\operatorname{diam} T \le 2 \cdot \operatorname{diam} G$

not true for every sp.

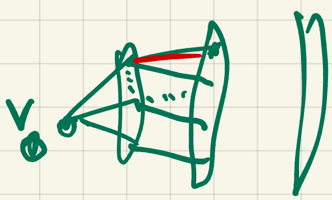
find example where

$\operatorname{diam} T$ much larger than $\operatorname{diam} G$



$\operatorname{diam} G = 2$

P_n is a sp. tree $\operatorname{diam} = n-1$



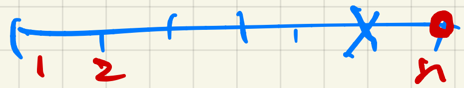
layer graph

by dist from v_0

BFS breadth-first search

14.82 count indep. sets in P_n

$\# = h(n)$



LEMMA $h(n) = h(n-1) + h(n-2)$

I_n set of indep sets: $I_n = \{$those that include $n\} \cup \{$don't$\}$

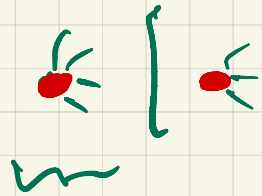
$|A_n| = h(n-2)$ $\qquad I_n = A_n \dot\cup B_n$

$|B_n| = h(n-1)$ $\qquad$ LEMMA $\Rightarrow$ THM $h(n) = F_{n+2}$ ind.

14.85 $\alpha(G) \geq \frac{n}{\Delta+1}$ $\Delta = deg_{max}$

Pf: $\begin{cases} \chi(G) \leq \Delta+1 & \text{greedy coloring} \\ \alpha(G)\cdot\chi(G) \geq n & \text{another ex.} \end{cases}$

Sol #2 : greedy selection



$\leq \Delta+1$ $\geq \Delta+1$

repeat $\left\lceil \frac{n}{\Delta+1} \right\rceil$ times

14.88 max $\frac{\alpha(G)}{\beta(G)} = n-1$

smallest maximal indep set

"maximize numerator & minimize denominator" } X

max $\alpha(G) = n \longleftrightarrow G = \overline{K_n}$ $\frac{\alpha}{\beta} = 1$

min $\beta(G) = 1 \longleftarrow$ 

for $G \neq \overline{K_n}$ max $\alpha(G) = n-1$

$\boxed{\max \frac{\alpha}{\beta} \leq n-1}$ actually equal

14.117

$\chi \le 1+\Delta$ $\quad \Delta = \deg_{max}$

$\max \dfrac{1+\Delta}{\chi}$

$\min \chi = 1$

$\max(1+\Delta) = n$

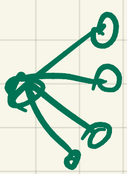
$\chi = 1 \iff G = \overline{K_n}$

$\Delta = 1 \qquad \dfrac{1+\Delta}{\chi} = 2$

If $G \neq \overline{K_n} \implies \chi \ge 2 \implies$

$$\boxed{\frac{1+\Delta}{\chi} \le \frac{n}{2}}$$

Claim $\frac{n}{2}$ is attainable

Pf <u>star</u> 

14.167 odd grid has no Hamilton cycle

(1) grid bipartite

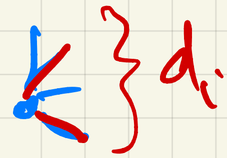
(2) $\hookrightarrow$ no odd cycle

if $kl = \text{odd}$: H-cycle would be an odd cycle $\Longrightarrow \nexists$

15.48 counting path of length 2 : $P_3(G)$

closed form in terms of $S = \sum d_i^2$, n , m

$$P_3(G) = \sum \binom{d_i}{2} = \frac{S}{2} - m$$



$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!}$$

k small

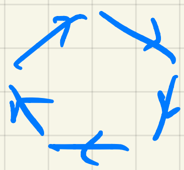
$$\binom{n}{2} = \frac{n(n-1)}{2} = \frac{n!}{2!(n-2)!}$$

(TRUE but don't)

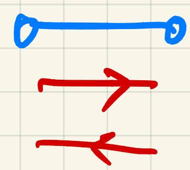
16.22 Count acyclic orientations of C_n, K_n

$\#(C_n) = 2^n - 2$

$m = n$



\# orient : 2^m



$\#(K_n)$

LEMMA: $\exists$ bijection between acyclic orientations of K_n and permutations of $[n]$

given by top. sort

$\therefore \; \# \quad n!$

$n \geq 2$

16.27 str conn digraph has $\geq n$ edges

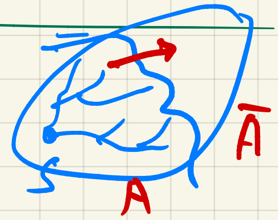
<u>Obs.</u> If G str conn, $n \geq 2$ $\Big\} \Rightarrow (\forall x \in V)(\deg^+(v) \geq 1)$

$\therefore \; n \leq \sum \deg^+(v) \overset{\text{directed handshake}}{=} m$

$s,t \in V \quad s \neq t$

16.40 (1) t is accessible from s

(2) $\forall (s,t)$-cut A $\quad |E(A,\bar{A})| \geq 1$



Claim (1) $\Leftrightarrow$ (2)

$\forall A$

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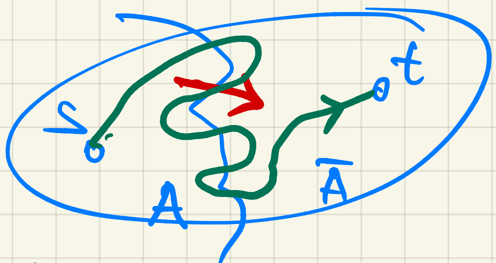
(1) $\Rightarrow$ (2)



P : $s \to \dots \to t$ path

$s = x_0 \to x_1 \to \dots \to x_k = t$

$x_0 \in A$
$x_k \in \bar{A}$

$j = \min \underbrace{\{ i \mid x_i \in \bar{A} \}}_{\neq \emptyset}$

$\left[\begin{array}{l} x_j \in \bar{A} \\ x_{j-1} \in A \end{array} \right.$

(s,t) cut $\quad x_{j-1} \to x_j$

(2) $\Rightarrow$ (1)

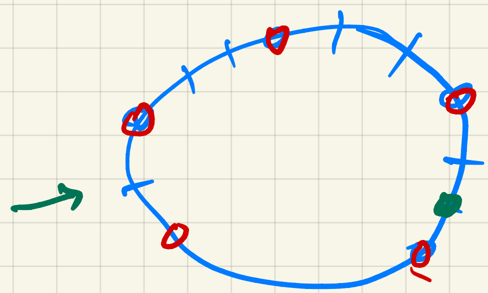
if t not acc. from s then $(\exists A)(E(A,\bar{A}) = \emptyset)$

$A := \{ x \in V \mid x \text{ is accessible from } s \}$

14.91 $k(n)$: size

smallest maximal indep set in C_n

$k(n) \le \left\lceil \frac{n}{3} \right\rceil$



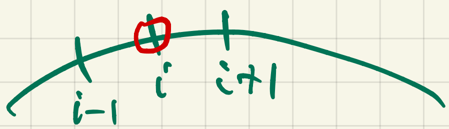
once we showed that this is a maximal indep set

indep ✓

maximal ✓

NTS $\quad k(n) \ge \left\lceil \frac{n}{3} \right\rceil \quad$ i.e. $\quad k(n) \ge \frac{n}{3}$

Pf:



mod n

S: max indep set

$\Rightarrow \forall i$

$i-1$ or i or $i+1 \in S$

indicator fctn of S: $\quad x_i = \begin{cases} 1 & i \in S \\ 0 & i \notin S \end{cases}$

$\forall i \quad x_{i-1} + x_i + x_{i+1} \ge 1 \quad \overset{?}{\Longrightarrow} \quad |S| \ge \frac{n}{3}$

Subscripts: mod n

$|S| = \sum x_i$

$$\left\{ \begin{aligned} x_1 + x_2 + x_3 &\ge 1 \\ x_2 + x_3 + x_4 &\ge 1 \\ &\vdots \\ x_{n-1} + x_n + x_1 &\ge 1 \\ x_n + x_1 + x_2 &\ge 1 \end{aligned} \right\} \text{ n inequalities}$$

add them up

$3|S| \ge n$ ✓

14.152: find a triangle-free graph of $\chi = 4$

$n = 11$ rotational symm of order 5

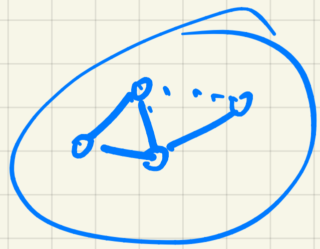
$\chi = 4$ easy: $\chi \le 4$ 4-colorable

essence: $\chi > 3$ not 3-colorable

15.25 $G \in \mathcal{G}(n,p)$ X = # triangles

$X = \sum Y_i$ Y_i : indicator of i^{th} triangle in K_n being present in G

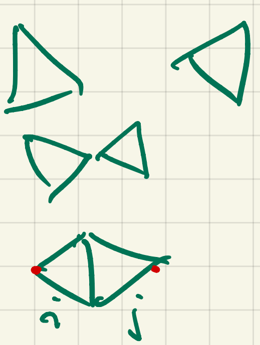
$i = 1, \dots, \binom{n}{3}$



$E(Y_i) = p^3$

$Var(Y_i) = p^3(1-p^3)$

$$\operatorname{Var} X = \sum_i \operatorname{Var} Y_i + \sum_{i \neq j} \operatorname{Cov}(Y_i, Y_j) = \binom{n}{3} \cdot p^3(1-p^3) + \binom{n}{4} \cdot 12 \cdot p^5(1-p)$$



indep $\Rightarrow$ Cov $= 0$

— " —

NOT

$$\operatorname{Cov}(X_i, X_j) = \underbrace{E(X_i X_j)}_{p^5} - \underbrace{E(X_i)E(X_j)}_{p^6} = p^5(1-p)$$

$\binom{n}{4} \cdot 4 \cdot 3$

$$\sim \frac{n^4 \cdot 12}{4!} p^5(1-p) = \frac{n^4}{2} p^5(1-p)$$

16.90 random tournament

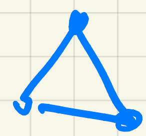
$\underbrace{E(\#\text{ oriented cycles})}_{X}$

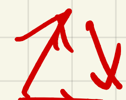
$\rightarrow \quad \frac{1}{2}$

$\leftarrow \quad \frac{1}{2}$

$X = \sum Y_i \qquad i = 1, \dots, \binom{n}{3}$

triangles in K_n





no

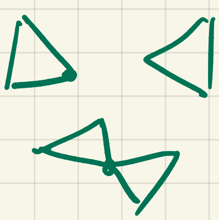
yes

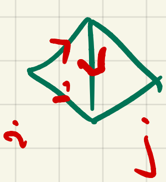
$E(Y_i) = \frac{2}{8} = \frac{1}{4}$

$\mathrm{Var}(Y_i) = \frac{1}{4} \cdot \frac{3}{4} = \frac{3}{16}$

Lemma The Y_i are pairwise indep !

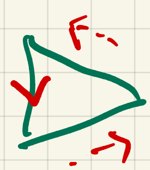
$\therefore \quad \mathrm{Var}\, X = \sum \mathrm{Var}\, Y_i = \binom{n}{3} \cdot \frac{3}{16} = \frac{n(n-1)(n-2)}{32}$





given orientation of i^{th} triangle

$P(j^{th}$ triangle is $\vec{C}_3) = \frac{1}{4}$



condition did not change prob.