2022-10-06

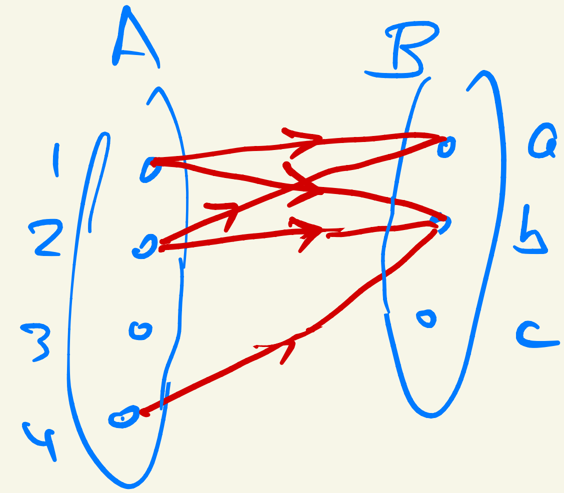
p1

A, B: sets

Relation from A to B

$$R \subseteq A \times B$$

if $(a,b) \in R$ we write aRb



Ex $\quad R = "<" \qquad (5,7) \in "<"$

$5 < 7$

$1Rb$

$A = \operatorname{dom}(R)$

$B = \operatorname{codom}(R)$

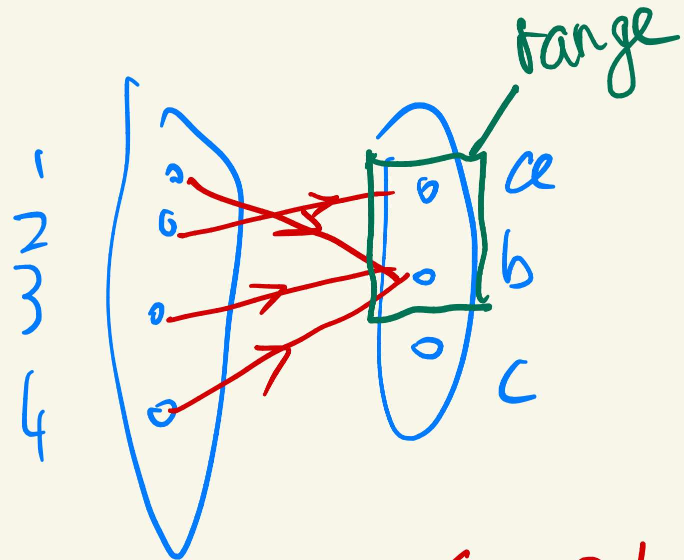
R is a function $A \to B$

if

$$(\forall x \in A)(\exists!\, y \in B)(xRy)$$

↑ unique

$y = R(x)$



$\operatorname{Range}(R) = \{y \in B \mid (\exists x \in A)(R(x) = y)\}$

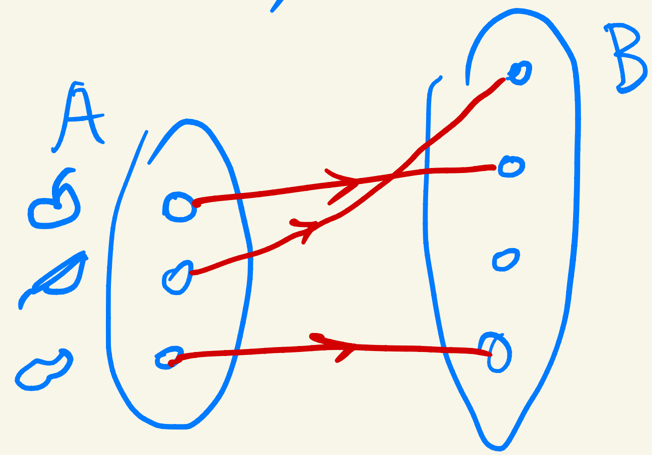
Injection $f: A \to B$

is a function s.t.

$$(\forall x, y \in A)(f(x) = f(y) \Rightarrow x = y)$$

$|A| = k \qquad |B| = \ell$

\# $A \to B$ injections



example $|A| = 3, \quad |B| = \ell$

$$\ell(\ell-1)(\ell-2)$$

x	$f(x)$
	ℓ
	$\ell-1$
	$\ell-2$

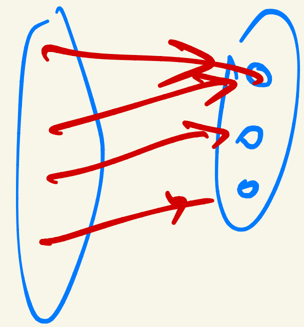
general case

$$\underbrace{\ell(\ell-1)\cdots(\ell-k+1)}_{k \text{ terms}}$$

function

Surjection

Range = codomain



$f : A \xrightarrow[\text{onto}]{} B$

DEF The function $f : A \to B$ is a <u>surjection</u> if

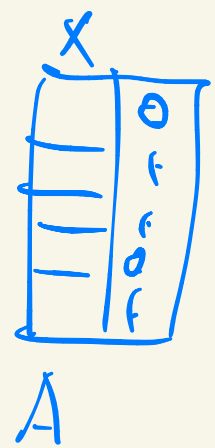
$$(\forall y \in B)(\exists x \in A)(f(x) = y)$$

Ex. Count $f : A \to B$ surjections when $|A| = k$, $|B| = 2$

functions $A \to B$: 2^k

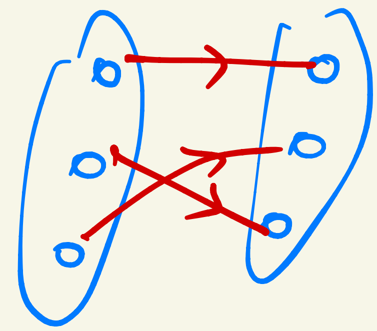
non-surjections : 2

$\therefore$ # Surj $= 2^k - 2$



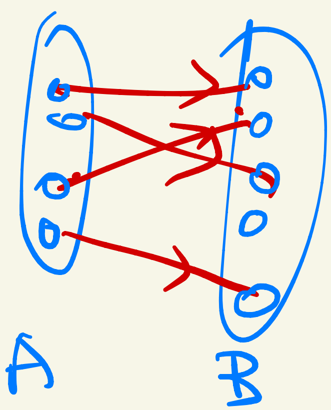
$B = \{0, 1\}$

$\boxed{\text{HW}}$ $|B| = 3$



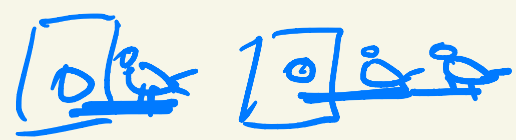
DEF f is a <u>bijection</u> if
f is both <u>inj</u> and <u>surj</u>

another name: 1-to-1 correspondence
between A and B



(*) If $\exists f: A \to B$ injection
then $|A| \leq |B|$

If $\exists f: A \to B$ surj.
then $|A| \geq |B|$



(*) is the same as
If $|A| > |B|$ then $\nexists$ inj. $A \to B$
i.e. $(\forall f: A \to B)(\exists \text{ collision})$

PIGEON HOLE
PRINCIPLE

$|A| = |B| \iff \exists f : A \to B$ bijection

$\boxed{\text{HW}}$ $|\mathcal{P}(A)| = |\underbrace{\{0,1\}^k}_{\text{set of (0,1)-strings of length } k}|$ ← find bijective proof

$|A| = k$

$\downarrow$

2^k

$R \subseteq A \times B$ $\quad R^{op} = \{(b,a) \mid (a,b) \in R\} \subseteq B \times A$

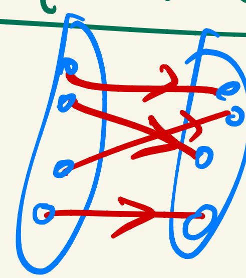
reversing all arrows

$f \subseteq A \times B$

f is a bijection $\iff$ both f and f^{op} are functions

---

f^{op} inverse function: f^{-1}



$|A| = n$

$Ev = \{\text{even subsets}\}$

subsets of even size

$Od = \{\text{odd subsets}\}$

$\boxed{HW}$ $|Ev| = |Od|$ unless $n = ?$

find simple bijection

"Proof from the book" (PAUL ERDŐS)
1913–1986

RELATIONS on A $\qquad R \subseteq A \times A$

(1) reflexive ? $\qquad (\forall a)(aRa)$

(2) symmetric ? $\qquad (\forall a,b)(aRb \Rightarrow bRa)$

(3) transitive ? $\qquad (\forall a,b,c)(aRb \wedge bRc \Rightarrow aRc)$

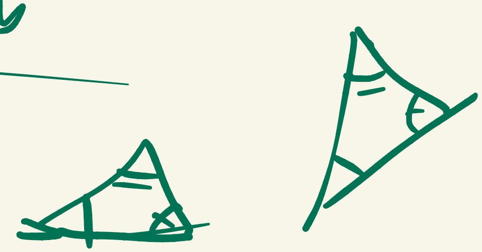
DEF rel. is an <u>equivalence relation</u> if $(1) \wedge (2) \wedge (3)$

Examples

- congruence mod m $\qquad$ (m fixed)
- identity relation: $R = \{(a,a) \mid a \in A\}$

  i.e. $\quad aRb \Leftrightarrow a=b$

- Similarity of triangles



Partition of A

LaTeX: Π : \Pi

is a set $\Pi = \{B_1, \ldots, B_k\}$ of "blocks"

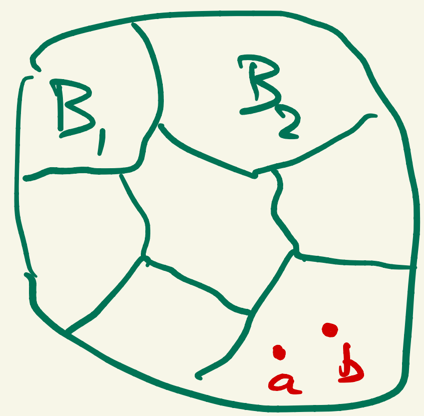
where $B_i \subseteq A$

$B_i \neq \emptyset$

if $i \neq j$ then $B_i \cap B_j = \emptyset$

(blocks are disjoint)

$$\bigcup_{i=1}^{k} B_i = A$$



---

defines a relation "belonging to the same block"

$$\sim_\Pi = \{(a,b) \mid a,b \in A \wedge (\exists i)(a,b \in B_i)\}$$

$(\exists B \in \Pi)(a,b \in B)$

$\sim_\Pi$ is an equiv. rel.

THM Fundamental thm of eq. relations

$(\forall$ eq. rel. R on $A)(\exists!$ partition Π of $A)(R = \sim_\Pi)$
