DEF

Relation on Ω \Omega

$$R \subseteq \Omega \times \Omega$$

Notation: if $(a,b) \in R$ we write aRb

ex $a < b$

$$\text{"}<\text{"} = \{(a,b) \mid a,b \in \mathbb{Z},\ a<b\}$$

DEF A relation $R \subseteq \Omega \times \Omega$ is an equivalence relation if

R is

reflexive

symmetric — $(\forall a,b \in \Omega)(aRb \Rightarrow bRa)$

transitive

~~if $\exists aRb$ then bRa~~

$(\exists a, b \in \Omega)(aRb)$ means $R \neq \emptyset$

R is <u>not</u> symmetric if and only if $(\exists a, b \in \Omega)(aRb \wedge \neg bRa)$

DEF R is antisymmetric if $(\forall a, b)(aRb \Rightarrow \neg bRa)$ $\boxed{b \not R a}$

negating an implication

$\neg(A \Rightarrow B)$ means $\underline{A \wedge \neg B}$

Given Ω, does there exist a relation that is both <u>symm</u> and <u>antisymm</u>

YES: $\emptyset$ $\boxed{\text{Do}}$: this is the only example

---

DEF **Partition of Ω**

<u>term</u>: B_i are blocks

$$\Pi = \{B_1, \dots, B_k\}$$

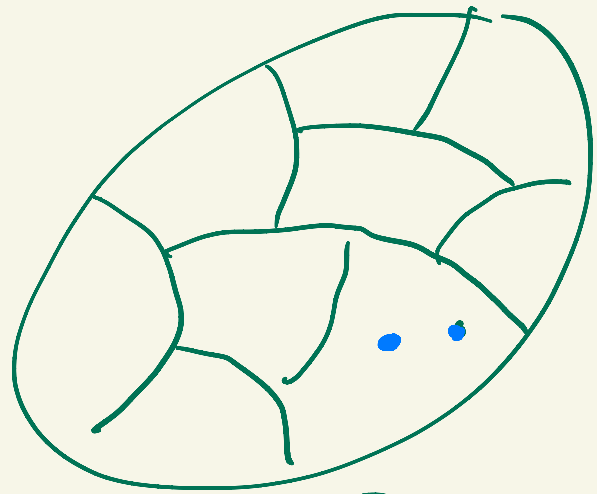
(1) $B_i \subseteq \Omega$

(2) $B_i \neq \emptyset$

disjoint

(3) if $i \neq j$ then $B_i \cap B_j = \emptyset$

(4) $\bigcup_{i=1}^{k} B_i = \Omega$

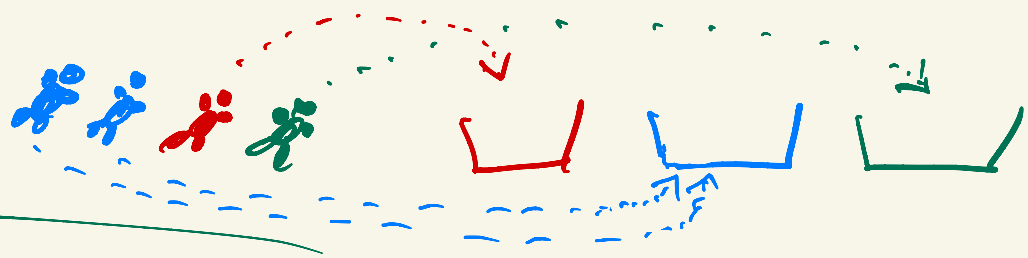


Given Π define corresponding eq. rel. on Ω

notation: $\sim_\Pi$ for $a, b \in \Omega$ we say $a \sim_\Pi b$ if $(\exists i)(a, b \in B_i)$

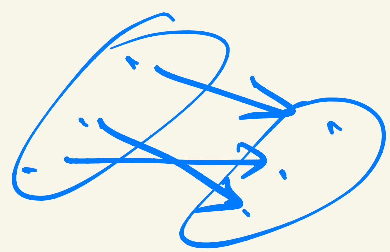
Fundamental Thm of Equiv. Relations

$$(\forall \text{ eq.rel } R \text{ on } \Omega)(\exists! \text{ partition } \Pi \text{ of } \Omega)(R = \sim_\Pi)$$

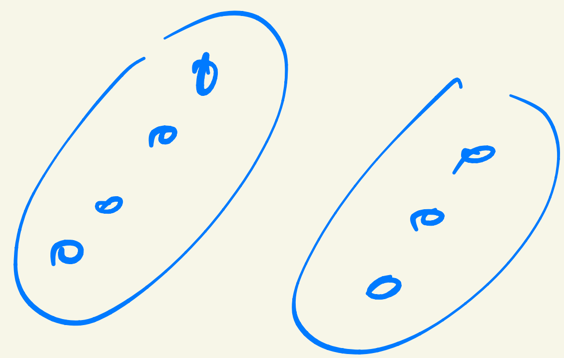


blocks of this partition are called the equivalence classes

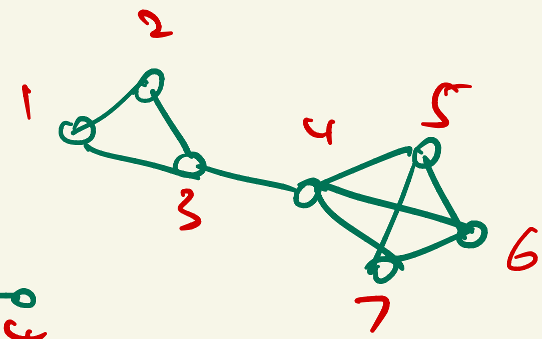
instances of the concept!



bijection

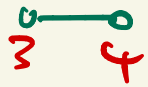


$\{3,1,3,3,1\}=\{1,3\}$

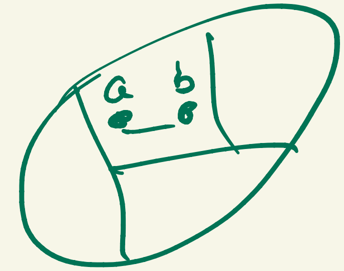


Clustering problem

remove



$\Longrightarrow$ equiv. rel.



Proof Given an eq rel R, find corresp. partition

1. $a \in \Omega$ $\quad [a] = \{ b \in \Omega \mid aRb \}$
2. need to show:

$$\Psi = \{ [a] \mid a \in \Omega \} \text{ partition } \Omega$$

2a $\quad [a] \subseteq \Omega$

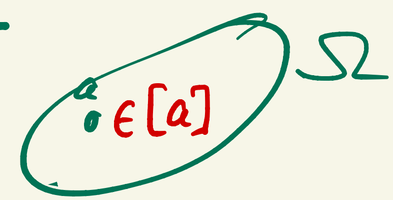
2b $\quad [a] \neq \emptyset$ b/c $a \in [a]$ b/c R is reflexive

2c $\quad (\forall a,b)([a]=[b] \lor [a] \cap [b] = \emptyset)$ $\leftarrow$

2d $\quad \bigcup_{a \in \Omega} [a] = \Omega$

To help intuition:
If $R = \sim_\pi$
then $[a]$ is the block containing a

$[a]=[b] \iff aRb$



DEF $a \in \bigcup_{i=1}^{m} C_i$ if $(\exists i)(a \in C_i)$

set

$\gcd(12, 18) = 6$

back to $\mathbb{Z}$

DEF $\gcd(a,b)$

We say that $d = \gcd(a,b)$ if ..

(1) $d \mid a \wedge d \mid b$ common divisor

(2) $d = \max \mathrm{Div}(a,b)$

$\mathrm{Div}(a) = \{c : c \mid a\}$

$\mathrm{Div}(a,b) := \mathrm{Div}(a) \cap \mathrm{Div}(b)$

$\gcd(0,0)$?

. . . . .

$\mathrm{Div}(0) = \mathbb{Z}$

$\mathrm{Div}(0,0) = \mathbb{Z} \cap \mathbb{Z} = \mathbb{Z}$