# Sequences

$a_0, a_1, a_2, \dots$

$a : \mathbb{N}_0 \to \mathbb{R}$

$\mathbb{C}$

## Geometric progression with quotient $q$:

recurrence: $a_n = q \cdot a_{n-1} \quad (n \geq 1)$

initial value: $a_0$

$a_0, a_0 q, a_0 q^2, \dots$

explicit formula: $a_n = a_0 \cdot q^n$

$$0^n = \begin{cases} 0 & \text{if } n \geq 1 \\ 1 & \text{if } n = 0 \end{cases} \qquad n \geq 0$$

Fibonacci sequence

initial values $F_0 = 0$

$F_1 = 1$

(*) recurrence $F_n = F_{n-1} + F_{n-2} \quad (n \geq 2)$

$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$

ORD $(\forall n \geq 1)(\gcd(F_{n-1}, F_n) = 1)$

Consecutive Fib. numbers are relatively prime

DEF $g_0, g_1, \ldots,$ is a Fibonacci-type sequence if it satisfies the Fibonacci recurrence (*)

HW Find all values of $q$ s.t. a geom. progr. with quotient $q$ is Fib. type

# Pascal triangle

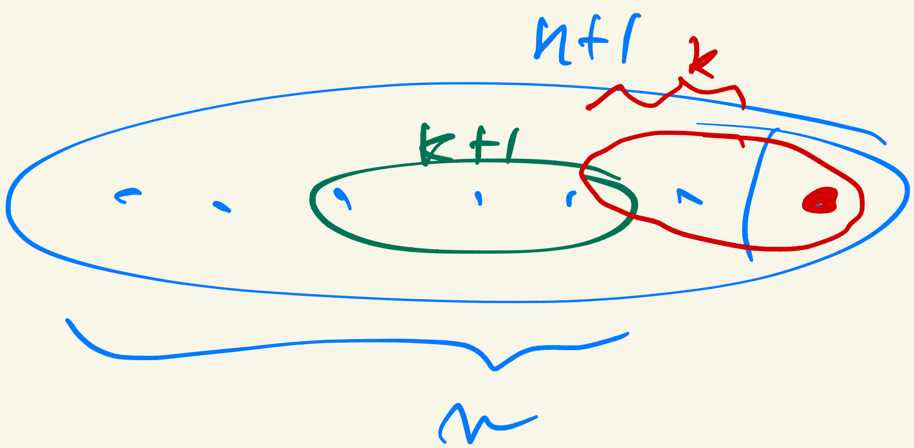
$$\binom{0}{0}$$
$$\binom{1}{0} \quad \binom{1}{1}$$
$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$
$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

$$\binom{n}{0} = 1$$
$$\binom{n}{n} = 1$$

$$\boxed{\phantom{x}}$$

## PASCAL'S IDENTITY

$$\binom{n+1}{k+1} = \underbrace{\binom{n}{k}}_{A} + \underbrace{\binom{n}{k+1}}_{B} \quad \checkmark$$




A: # (k+1)-subsets s.t.
it contains the special pt

B: " — s.t.
it does not contain sp. pt.

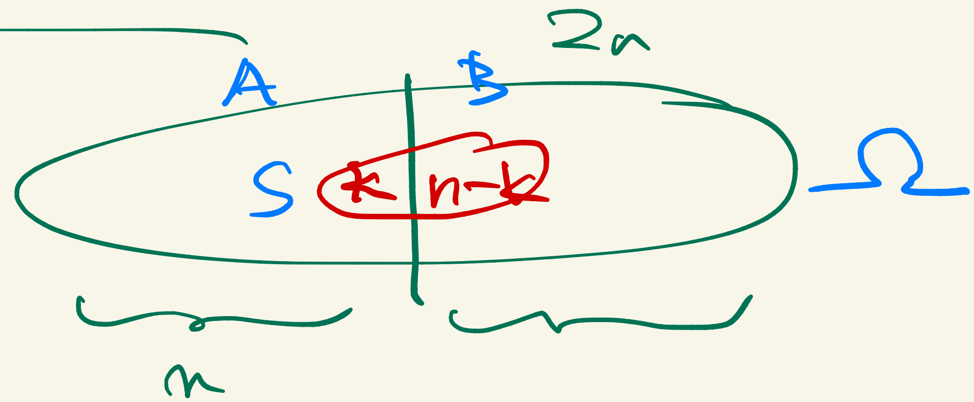
# Vandermonde identity

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

**Notation:**

$$\binom{\Omega}{k} = \{k\text{-subsets of } \Omega\} = \{R \subseteq \Omega \mid |R| = k\}$$

**Proof**




$|\Omega| = 2n$

$\Omega = A \,\dot\cup\, B$ ← disjoint union: $A \cap B = \emptyset$

$|A| = |B| = n$

$k = 0, \dots, n$

$$T_k = \{S \subseteq \Omega \mid |S \cap A| = k,\ |S \cap B| = n-k\}$$

$$\left|\binom{\Omega}{k}\right| = \binom{|\Omega|}{k}$$

$$T_k \subseteq \binom{\Omega}{n}$$

$$|T_k| = \binom{n}{k} \cdot \underbrace{\binom{n}{n-k}}_{\binom{n}{k}} = \binom{n}{k}^2$$

$$\binom{\Omega}{n} = \dot{\bigcup_{k=0}^{n}} T_k \quad \therefore \quad \left|\binom{\Omega}{n}\right| = \sum |T_k|$$

disjoint union

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2 \quad \checkmark$$

So $\{T_0, T_1, \dots, T_n\}$ is a partition of $\binom{\Omega}{n}$

(*) $\boxed{\sum_{k=0}^{n}\binom{n}{k} = 2^n}$

combinatorial proof

$|\Omega| = n$

$\mathcal{P}(\Omega) = \dot{\bigcup}_{k=0}^{n}\binom{\Omega}{k}$

$2^n = \sum_{k=0}^{n}\binom{n}{k}$

## Binomial Theorem

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \dots + \binom{n}{n}y^n$$

$$= \sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k}$$

"Algebra proof" of (*): use Binomial Thm

$x := y := 1$ $\quad 2^n = \sum \binom{n}{k}\underbrace{1^k \cdot 1^{n-k}}_{1}$ ✓

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$ $\Leftarrow$ Binomial Thm most useful special case

$$S(n,2) = \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n}{2j} = \sum_{j=0}^{\infty} \binom{n}{2j}$$

?

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \cdots = 2^n$$

$\lfloor x \rfloor$ : floor of $x$

rounded down value

$\lfloor 5.3 \rfloor = 5$

$\lfloor -5.3 \rfloor = -6$

Conjecture $S(n,2) = \frac{2^n}{2} = 2^{n-1}$ for $n \geq 1$ $S(0,2) = 1$

$$\sum_{k=0}^{\infty} \binom{n}{2k} = E_n \qquad \text{\# even subsets of } [n]$$

$$\sum_{k=0}^{\infty} \binom{n}{2k+1} = O_n \qquad \text{\# odd} \quad ''$$

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$$E_n + O_n = 2^n$$

$$E_n - O_n = \begin{cases} 1 & \text{if } n=0 \\ 0 & \text{if } n \geq 1 \end{cases}$$

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$$2E_n = \begin{cases} 2^0+1=2 & \text{if } n=0 \\ 2^n & \text{if } n \geq 1 \end{cases}$$

$$E_n = \begin{cases} 1 & \text{if } n=0 \\ 2^n/2 & \text{if } n \geq 1 \end{cases}$$

Combinat. proof

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Algebra proof of $E_n - O_n = \begin{cases} 1 & n=0 \\ 0 & n \geq 1 \end{cases}$

$$E_n - O_n = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + - \cdots$$

$$=$$

Binomial Thm $(1+x)^n = \cdots$ set $x=-1$

$$E_n - O_n = (1-1)^n = 0^n = \begin{cases} 1 & \text{if } n=0 \\ 0 & \text{if } n \geq 1 \end{cases}$$

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$$S(n,3) = \sum_{k=0}^{\infty} \binom{n}{3k} = \binom{n}{0} + \binom{n}{3} + \binom{n}{6} + \cdots$$

$$S(n,3) \overset{?}{=} \frac{2^n}{3}$$ ← never an integer by FTA

$\boxed{\text{CH}}$ $\left| S(n,3) - \frac{2^n}{3} \right| < 1$

# REVIEW LIMITS