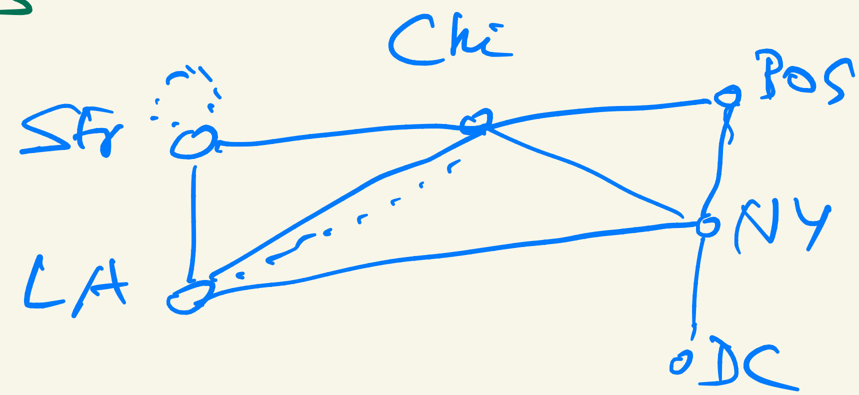
GRAPH THEORY

graph : nodes and links

$\Omega = \{LA, SFr, Chi, Bos, NY, DC\}$

links = {{LA, SFr}, {LA, Chi}, ...}



airline routes

irreflexive

symmetric

not transitive (unless there are no links)

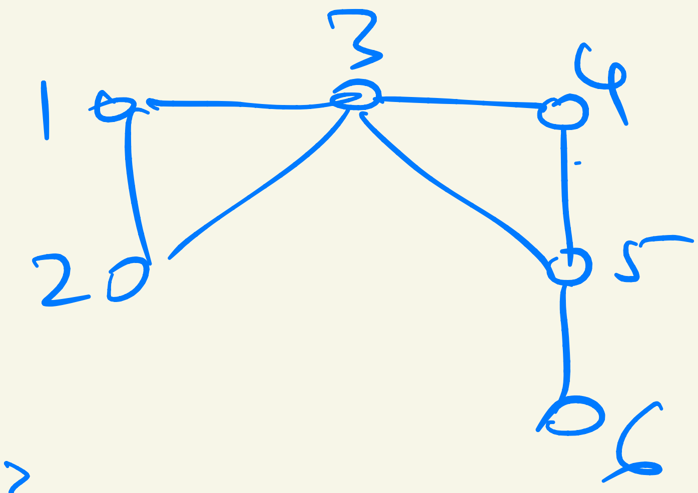
GRAPH $G = (V, E)$ singular:

V - set of vertices (cities) vertex

E - set of edges (links)

$V = [6]$

$E = \{\{1,2\}, \{1,3\}, \{2,3\}, \ldots\}$



edge: unordered pair

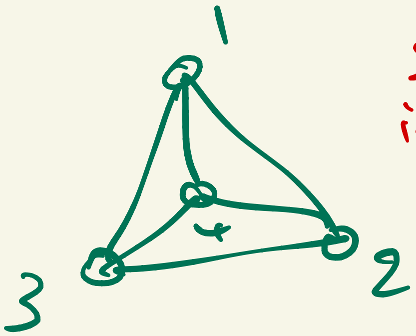
$E = \{12, 13, 23, 34, 35, 45, 56\}$

If $\{i,j\} \in E$ the we say i and j are ADJACENT

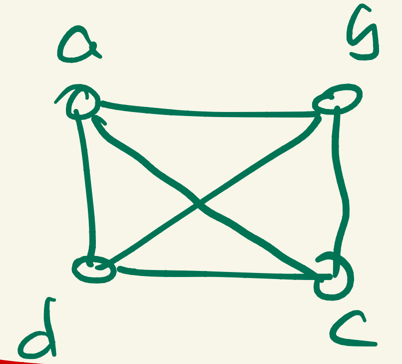
ADJACENCY relation $\sim$ on V $i \sim_G j$

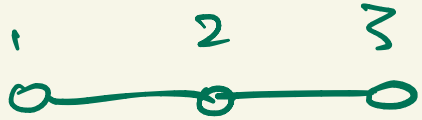
- irreflexive
- symmetric

$i \sim j$

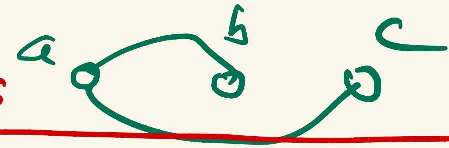


24 isomorphisms

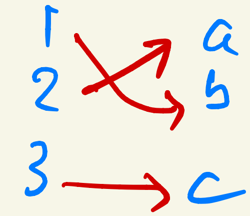


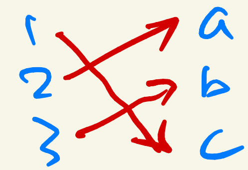


2 iso-morphisms



two isomorphisms





$G = (V, E)$ $\qquad H = (W, F)$

DEF An <u>isomorphism</u> from G to H is a <u>bijection</u> $f : V \longrightarrow W$ that <u>preserves</u> adjacency, i.e.

$$(\forall u, v \in V)\left(u \underset{G}{\sim} v \iff f(u) \underset{H}{\sim} f(v)\right)$$

DEF G and H are <u>isomorphic</u> if $\exists f : G \to H$ <u>isomorphism</u>

$G \cong H$

If $f: G \to H$ is an isomorphism
then $f^{-1}: H \to$ — " —

What kind of relation on graphs
is "isomorphism" ("being isomorphic")?

equivalence relation

$\Omega = \{\text{all graphs}\}$

$G \cong G$ identity: $(\forall x)(f(x) = x)$

if $G \cong H$ then $H \cong G$

if: $\exists f$ s.t. ...
then f^{-1} will work

$\boxed{\text{DO}}$ if $\left.\begin{matrix} G \cong H \\ H \cong K \end{matrix}\right\}$ then $G \cong K$

if $i \sim j$ we say i, j are <u>neighbors</u>

$v \in V$ $\quad$ degree(v) = # neighbors $= \deg_G(v)$

deg $\quad$ 1 $\quad$ 2 $\quad$ 1

$\boxed{\text{DO}}$ If $f : G \to H$ is an <u>isomorphism</u>

then $(\forall v \in V(G))(\deg_G(v) = \deg_H(f(v)))$

$V(G)$ = set of vertices of G

("isomorphisms preserve the degrees")

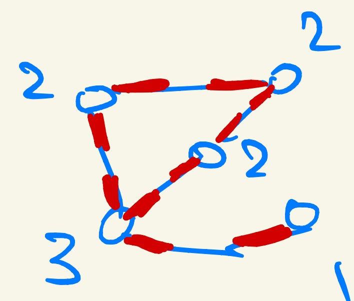
$\boxed{\text{DO}^{\Delta}}$ Converse false: $\exists\, G, H$, bijection $f : V(G) \to V(H)$ that preserves degrees <u>but</u> is <u>not</u> an isomorphism

degree sequence: deg(1), deg(2), ..., deg(n)
if $V = [n]$

NOTATION: $n = |V|$, $m = |E|$

$$\boxed{\sum_{v \in V} \deg(v) = 2m}$$

HANDSHAKE THEOREM



EX $\nexists$ G w 17 vertices
and all degrees = 3

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b/c that would mean $m = \frac{1}{2} \cdot 17 \cdot 3 \notin \mathbb{Z}$

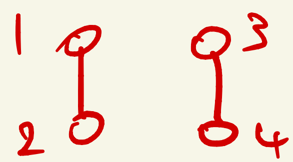
COROLLARY # vertices of odd degree is even

\# bijections $A \to A$ where $|A|=n$

$n!$

DO$^{\Delta}$: degree-preserving bijection is not necessarily an isomorphism

counterexample: $n=4$ $\quad 1,1,1,1$

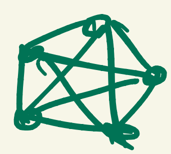


<u>DEF</u> G is <u>regular</u> of degree d if

$$(\forall v \in V)(\deg(v)=d)$$

<u>EXAMPLES</u>: <u>Complete graph</u> K_n

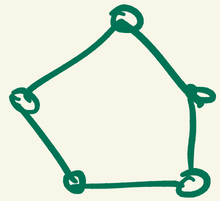
$\deg(v)=n-1$



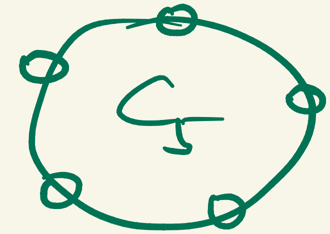
K_5

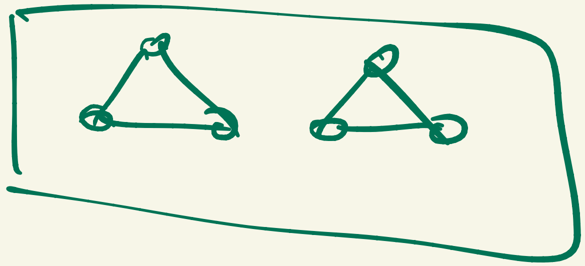
n-cycle

C_n



C_5





$n = 6$

$\deg = 2$

Ex Find G, H: (1) same degree sequence

(2) not isomorphic

HW

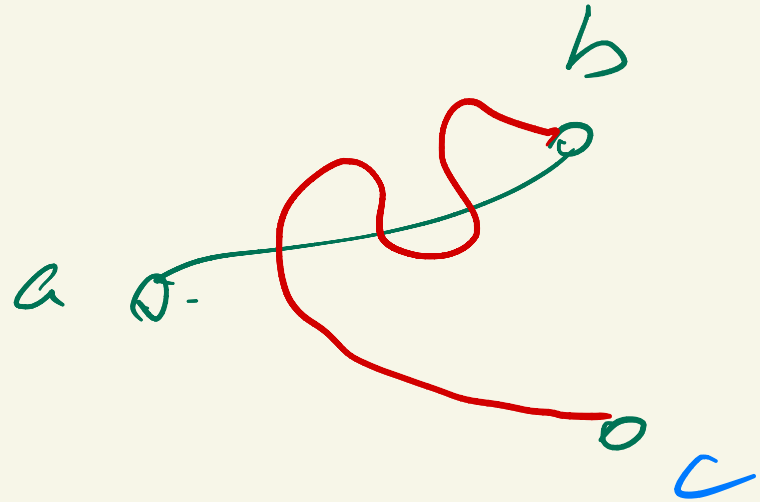
min ← n, min ← m for given n

path of length n

P_{n+1}

P_4

G is <u>connected</u> if $(\forall u,v \in V)(\exists\, u - \cdots - v \text{ path in } G)$



Concatenation of two paths is not necessarily a path

YET

the relation "$\exists$ u-...-v path"

is an equivalence relation on V

$\boxed{\text{DO}}$ Why?