Finite Probability Spaces → handout

ordered pair (Ω, P)

where Ω is a SET *

and P is a FUNCTION ** $f: \Omega \to \mathbb{R}$

* Ω is a non-empty finite set

** s.t. $(\forall a \in \Omega)(P(a) \geq 0)$

$$\sum_{a \in \Omega} P(a) = 1$$

} DEF probability distribution over Ω

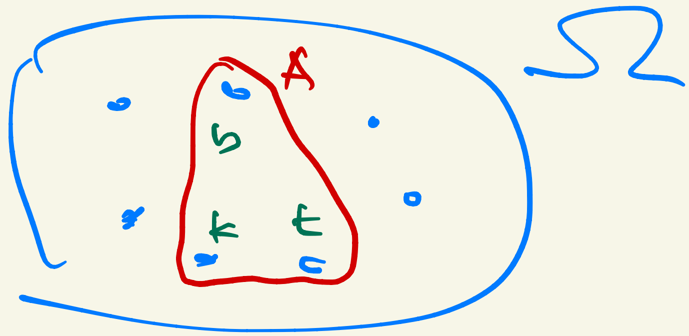
Ω is called "the sample space"

elements of Ω are called "elementary events"

Event: SUBSET OF Ω

~~DEF~~ If $A \subseteq \Omega$ is an event then we set

$$P(A) := \sum_{a \in A} P(a)$$



$P(A) = P(b) + P(k) + P(t)$

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in particular

$$P(\{a\}) = P(a)$$

If $|\Omega| = n$ then # events is 2^n

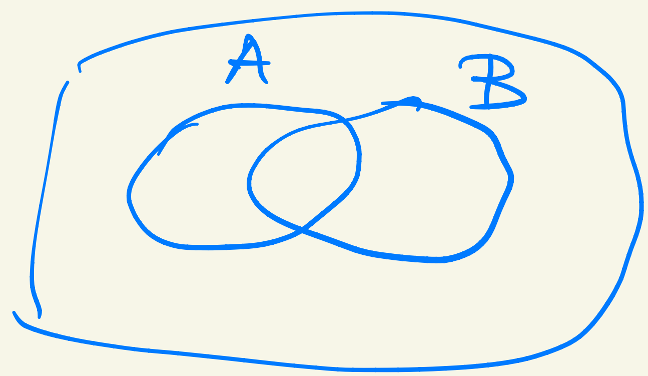
Conditional probability

Condition: $B \subseteq \Omega$ event   assumption: $\underline{\underline{P(B) \neq 0}}$

conditional probability: $P(A \mid B)$ $A \subseteq \Omega$

probability of A $\underline{\text{given } B}$

$$\boxed{P(A \mid B) := \frac{P(A \cap B)}{P(B)}}$$



DEF Events A and B are <u>independent</u>

if $P(A \cap B) = P(A) \cdot P(B)$

<u>Observation</u> If $P(B) \neq 0$ then

A, B indep $\iff P(A|B) = P(A)$

$\boxed{\text{DO}}$ If B is a trivial event $[P(B) = 0$ or $1]$ then $(\forall A \subseteq \Omega)(A$ and B are independent$)$

COROLLARY to Obs: If $P(A) \neq 0$, $P(B) \neq 0$ then

$P(A|B) = P(A) \iff P(B|A) = P(B)$

A, B indep. events if $P(A \cap B) = P(A) \cdot P(B)$

A, B positively correlated if $P(A \cap B) > P(A) \cdot P(B)$

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negatively correlated " $<$ "

$\boxed{\text{DO}}$ If $P(B) \neq 0$ then

A, B pos. cor. $\iff P(A \mid B) > P(A)$

neg. — " — $<$ — " —

DEF Events A, B, C are <u>independent</u> if

(i) A, B, C are <u>pairwise indep</u>

i.e. A, B are indep $\wedge$ A, C are ind.
$\wedge$ B, C are ind

(ii) $P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C)$

"independent" $\Longleftrightarrow$ " $\left.\begin{array}{l}\text{fully}\\ \text{mutually}\end{array}\right\}$ indep."

$\boxed{\text{HW}}$ (i) $\not\Rightarrow$ (ii) $\qquad$ min $\leftarrow |\Omega|$

$\boxed{\text{EX}}$ (ii) $\not\Rightarrow$ (i)

INDEPENDENCE OF k events

$A_1, \dots, A_k$ are indep. if

$$(\forall I \subseteq [k])\left(P\left(\bigcap_{i\in I} A_i\right) = \prod_{i\in I} P(A_i)\right)$$

2^k conditions $-k-1$

If $I = \{i\}$ $\quad |I| = 1$

If $I = \emptyset$ $\quad |I| = 0$

$$\bigcup_{i\in\emptyset} A_i := \emptyset$$

Convention $\displaystyle\bigcap_{i\in\emptyset} A_i := \Omega$

Example: if $A_1 \dots A_k$ are indep.

balanced events $\quad P(A_i) = \frac{1}{2}$

then $P\left(\bigcap_{i=1}^{k} A_i\right) = \frac{1}{2^k}$ ← exponentially small as a fctn of $\underline{k}$

RANDOM VARIABLES

DEF A random variable on the prob. space (Ω, P) is a function $X : \Omega \to \mathbb{R}$

Ω	X
a	$X(a)$
b	$X(b)$
c	$X(c)$

EXPECTED VALUE (MEAN VALUE)

of r.v. X is

$$\boxed{E(X) = \sum_{a\in\Omega} X(a)\cdot P(a)}$$

weighted average

Special case of uniform space: $|\Omega| = n$

$(\forall a \in \Omega)(P(a) = \frac{1}{n})$

$$E(X) = \frac{\sum_{a\in\Omega} X(a)}{n}$$

AVERAGE

$x_1, \dots, x_n \in \mathbb{R}$

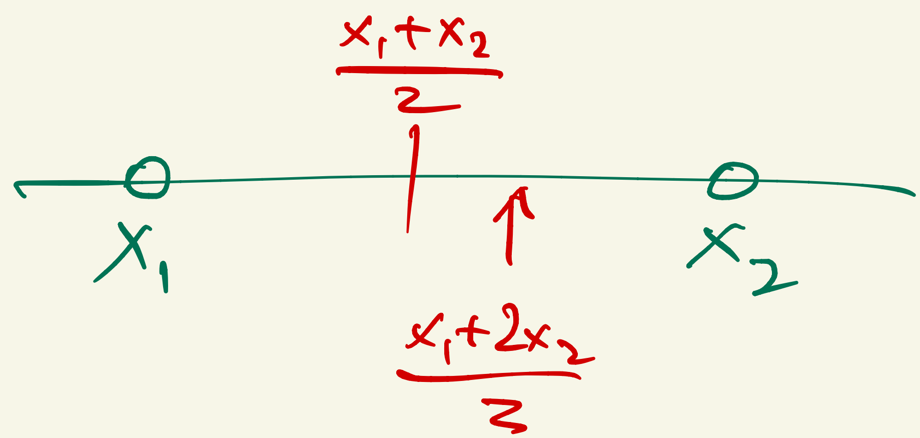
$p_1 \dots p_n$: prob. distr: $p_i \geq 0$

$\sum p_i = 1$

$A = \sum x_i p_i$ is a weighted average of the x_i

Example: $n = 2$ $\quad p_1 = \frac{1}{3}$ $\quad p_2 = \frac{2}{3}$

$$A = \frac{x_1 + 2x_2}{3}$$



weighted avg.

EX $\min x_i \le A \le \max x_i$

THM. Expected value: additive:

if $X_1 \dots X_n$ r.v.'s on same prob. space

then $E(\sum X_i) = \sum E(X_i)$