2022-11-08 | p1

PROBABILITY SPACE

(Ω, P) $\quad \Omega$ nonempty (finite) set "sample space"

P function $\Omega \to \mathbb{R}$

s.t. P is a probability distribution:

(i) $(\forall a \in \Omega)(P(a) \geq 0)$

(ii) $$\sum_{a \in \Omega} P(a) = 1$$

Event $A \subseteq \Omega$

extending P to $\mathcal{P}(\Omega)$: $\quad P(A) = \sum_{a \in A} P(a)$

RANDOM VARIABLE : function $\Omega \to \mathbb{R}$

EXPECTED VALUE of $X : \Omega \to \mathbb{R}$

$$E(X) = \sum_{a \in \Omega} X(a) \cdot P(a)$$

Obs this is a weighted average of the values of X

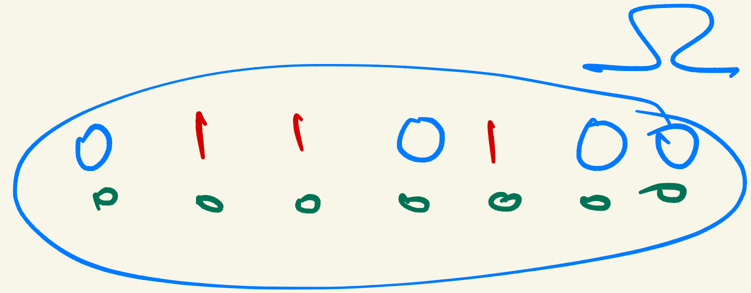
$$\therefore \quad \min X \leq E(X) \leq \max X$$

LINEARITY OF EXPECTATION

$$\boxed{\begin{array}{l} \text{If } X_1, \ldots, X_k : \Omega \to \mathbb{R} \text{ are r.v's} \\ c_1 \ldots c_k \in \mathbb{R} \text{ then} \\ E\left(\sum c_i X_i\right) = \sum c_i E(X_i) \end{array}}$$

Indicator variable : $X : \Omega \to \{0,1\}$

$$A = \{ a \in \Omega \mid X(a) = 1 \}$$



Conversely, given event $A \subseteq \Omega$

let

$$\vartheta_A : \Omega \to \{0,1\}$$

be defined by

$$\vartheta_A(a) = \begin{cases} 1 & \text{if } a \in A \\ 0 & \text{if } a \notin A \end{cases}$$

bijection between events and indicator variables

Thm If $A \subseteq \Omega$ then $\boxed{E(\vartheta_A) = P(A)}$

Event "$\vartheta_A = 1$" $= A$

Proof: $E(\vartheta_A) = 1 \cdot P(\vartheta_A = 1) + 0 \cdot P(\vartheta_A = 0) = P(A)$ ✓

Thm X r.v.

$$\Rightarrow \quad E(X) = \sum_{y \in \mathbb{R}} y \cdot P(X = y) = \sum_{y \in \mathrm{Range}(X)} (\text{same})$$

$$\text{"}X = y\text{"} = \{a \in \Omega \mid X(a) = y\}$$

Proof: combine terms with same value of X.

Experiment: flip n coins

$X = \#\ \underline{\text{heads}}$

range $= \{0, \dots, n\}$

$|\Omega| = 2^n$

$|\mathrm{range}(X)| = n+1$

Biased coin $\quad P(\text{heads}) = p$

$P(\text{tails}) = 1-p$

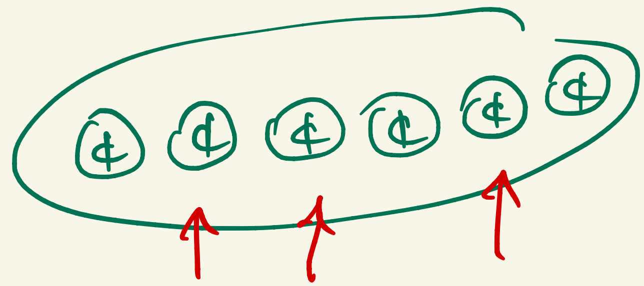
flip it n times

$X = \#\text{heads}$

$E(X) = ?$ $\quad$ intuitively $p \cdot n$

$$E(X) = \sum_{y=0}^{n} y \cdot P(X=y) = \sum_{y=0}^{n} y \cdot \binom{n}{y} p^y \cdot (1-p)^{n-y}$$

\# coin flip sequences with exactly y heads is $\binom{n}{y}$



$(1-p)\cdot p \cdot p \cdot (1-p) \cdot p \cdot (1-p)$

$p^y \cdot (1-p)^{n-y}$

$$\sum_{y=0}^{n} y\cdot\binom{n}{y} p^y(1-p)^{n-y} = \sum_{y=1}^{n} \cdots$$

$$y\binom{n}{y} = y\cdot\frac{n(n-1)\cdots(n-y+1)}{y\cdot(y-1)\cdots 1} = n\cdot\frac{(n-1)\cdots(n-y+1)}{(y-1)\cdots 1} = n\cdot\binom{n-1}{y-1}$$

$$= n\cdot\sum_{y=1}^{n}\binom{n-1}{y-1} p^y(1-p)^{n-y} =$$

$$\boxed{z := y-1} = np\sum_{z=0}^{n-1}\binom{n-1}{z}p^z\cdot(1-p)^{n-1-z} = np\cdot\big(\underbrace{p+(1-p)}_{1}\big)^{n-1} = np\cdot 1 = np$$

$$p^y = p^{z+1} = p\cdot p^z$$

binomial thm

Proof #2 Thm $E(X) = np$

X_i = indicator the i^{th} coin Heads $i = 1,\ldots,n$

$$X = \sum_{i=1}^{n} X_i \quad \therefore E(X) = \sum E(X_i) = \sum \underbrace{P(X_i = 1)}_{p} = \boxed{np}$$

n coins; $P(i^{th}$ coin Heads$) = p_i$

$X = \#$ heads $\qquad X = \sum X_i \qquad X_i$: indicator that i^{th} coin Heads

$$\boxed{E(X)} = \sum E(X_i) = \sum P(X_i = 1) = \boxed{\sum_{i=1}^{n} p_i}$$

"Bernoulli trial": indicator var.

1 "success"

0 "failure"

Independence of r.v's

$X, Y : \Omega \to \mathbb{R}$

are **indep** if $(\forall x, y \in \mathbb{R})$("$X=x$" and "$Y=y$" are indep.)

If $P(Y=y) \neq 0$ then this means the same as
$$P(X=x \mid Y=y) = P(X=x)$$

So X, Y indep. if $(\forall x, y \in \mathbb{R})(P(X=x \wedge Y=y) =$
$= P(X=x) \cdot P(Y=y))$

DEF

DEF $X_1, \ldots, X_k : \Omega \to \mathbb{R}$ r.v's

We say that the X_i are <u>independent</u> if

$$(\forall x_1 \ldots x_k \in \mathbb{R})\left(P\left(\bigwedge_{i=1}^{k} X_i = x_i\right) = \prod_{i=1}^{k} P(X_i = x_i)\right)$$

Thm If $(X_i \mid i \in I)$ are indep and $J \subseteq I$ then $(X_i \mid i \in J)$ also indep

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$\boxed{\text{DO}}$ If X, Y, Z are indep. r.v's then $e^X, \sqrt{Y^2+1}, \arctan Z$ indep

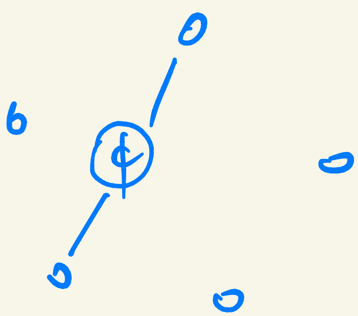
If X, Y, Z, U, V, W are indep r.v's

then $X+Y+e^Z$, $U\cdot\cos V$, $\arctan W$ — " —

Random graph

ERDŐS–RÉNYI 1960

$\mathcal{G}(n,p)$ n # vertices p: $P(\{i,j\}\in E)$



$$|\Omega| = 2^{\binom{n}{2}}$$

Set of vertices fixed, adjacency decided by independent coin flips