THM If X, Y independent r.v's then
$$E(XY) = E(X) \cdot E(Y)$$

pointwise multiplication: $(\forall a \in \Omega)$ $(XY)(a) := X(a) \cdot Y(a)$

$X_1, \ldots, X_k$ indep. r.v's then
$$E\left(\prod_{i=1}^{k} X_i\right) = \prod_{i=1}^{k} E(X_i)$$

X, Y indep $\Longrightarrow E(XY) = E(X)E(Y)$

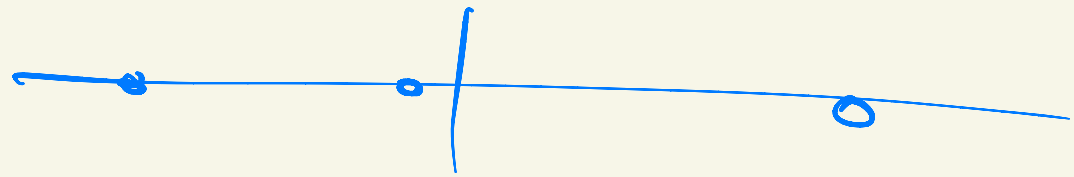
$\not\Longleftarrow$

$\boxed{\text{HW}}$

$\min \leftarrow |\Omega|$

WARNING: Define prob space first

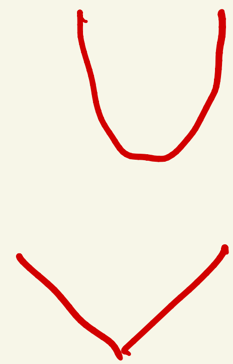
#1 aggregate param. of a r.v. $E(X)$



$E(X)$

#2 variance

$$Var(X) = E\left((X - E(X))^2\right)$$



x^2

$|x|$

standard deviation

$$\sigma(X) = \sqrt{Var(X)}$$

\sigma

let $f(x) = \cos x$

$$f^2(x) = \left(f(x)\right)^2 = (\cos x)^2$$

DEF $\operatorname{Var} X = E\left(\left(X - E(X)\right)^2\right) \geq 0$

thm $\operatorname{Var} X = E(X^2) - \left(E(X)\right)^2$

COROLLARY $E(X^2) \geq \left(E(X)\right)^2$

CAUCHY-SCHWARZ

DEF Covariance of r.v's X, Y

$$\operatorname{Cov}(X,Y) = E(XY) - E(X) \cdot E(Y)$$

$\therefore$ $\boxed{\operatorname{Cov}(X,X) = \operatorname{Var}(X)}$

$\therefore$ If X, Y indep then $\operatorname{Cov}(X,Y) = 0$

$\Longrightarrow$

$\not\Longleftarrow$

DEF X, Y are

positively correlated if $\mathrm{Cov}(X,Y) > 0$

uncorrelated if $\mathrm{Cov}(X,Y) = 0$

negatively correlated if $\mathrm{Cov}(X,Y) < 0$

TODO Events A, B are

pos. correl $\iff$ ϑ_A, ϑ_B pos. correl

independent $\iff$ ϑ_A, ϑ_B uncorrel. $\iff$ indep

neg. correl $\iff$ — " — neg. correl

indicator variables

$m = E(X)$ $\qquad$ $Var(X) = E((X-m)^2) =$

$$= E(X^2 - 2mX + m^2)$$

$$= E(X^2) - 2m\underbrace{\underbrace{E(X)}_{m}}_{-2m^2} + m^2 = E(X^2) - m^2$$

thm.

$$\boxed{Var(X) = E(X^2) - (E(X))^2}$$

$$\boxed{\mathrm{Cov}(X,Y) = \mathrm{Cov}(Y,X)}$$

Let $X = X_1 + \dots + X_k$

Thm

$$\mathrm{Var}(X) = \sum_{i=1}^{k} \sum_{j=1}^{k} \mathrm{Cov}(X_i, X_j)$$

$$= \sum_{i=1}^{k} \mathrm{Var}(X_i) + 2 \cdot \sum_{1 \le i < j \le k} \mathrm{Cov}(X_i, X_j)$$

$\mathrm{Cov}(X_i, X_i)$ "diagonal terms"

COR. If $X_1 \dots X_k$ are pairwise indep.

then $\mathrm{Var}(\sum X_i) = \sum \mathrm{Var}(X_i)$

Let $X = \sum_{i=1}^{k} X_i$ r.v's

unconditionally $E(X) = \sum E(X_i)$

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if the X_i are

pairwise indep then $Var(X) = \sum Var(X_i)$

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if the are

independent then $E(\prod X_i) = \prod E(X_i)$

Markov's inequality
Chebyshev's inequality $\Big\}$ STUDY $\leftarrow$ concentration ineq.

ASYMPTOTIC NOTATION

a_n, b_n sequences

DEF $a_n \sim b_n$ asymptotically equal if $\frac{a_n}{b_n} \to 1$

This is a tail event: not affected by changing a finite number of terms

all that comes after some threshold

DEF $a_n = o(b_n)$ a_n is little-oh of b_n if $\frac{a_n}{b_n} \to 0$

$\boxed{\text{HW}}$ $a_n \sim b_n \iff a_n - b_n = o(a_n)$

$\iff a_n - b_n = o(b_n)$

DEF

$$a_n = O(b_n)$$

a_n is big-Oh of b_n

if $(\exists C, n_0)(\forall n \geq n_0)(|a_n| \leq C \cdot |b_n|)$

Example

$$100n^2 + 1000n - 5 = O(n^2)$$

$C = 101$ works

what (smaller) constants work?

every $C > 100$

DEF

$$a_n = \Omega(b_n) \quad \text{big-Omega}$$

a_n is — of b_n

if $b_n = O(a_n)$

P 11

SAME RATE OF GROWTH

DEF $a_n = \Theta(b_n)$ a_n is big-Theta of b_n

if $a_n = O(b_n)$ and $a_n = \Omega(b_n)$

$$100n^2 + 1000n - 5 = \Theta(n^2)$$

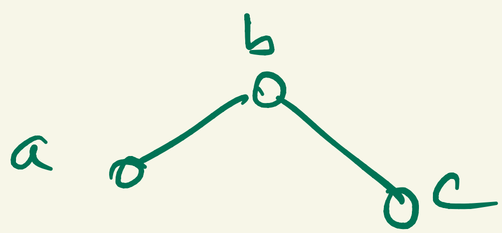
GRAPHS

$G = (V, E)$

where V, E are sets, $E \subseteq \binom{V}{2}$

adjacency relation on V:

$a \sim_G b$ if $\{a, b\} \in E$ — a, b are neighbors



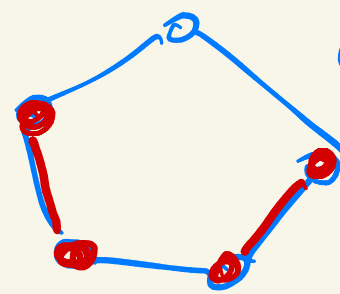
$$\binom{\Omega}{k} = \{ A \mid A \subseteq \Omega, |A| = k \}$$

$$\left| \binom{\Omega}{k} \right| = \binom{|\Omega|}{k}$$

SUBGRAPH

$H = (W, F)$ is a subgraph of $G = (V, E)$ if $W \subseteq V$ and $F \subseteq E$

DEF H is a spanning subgraph if $W = V$

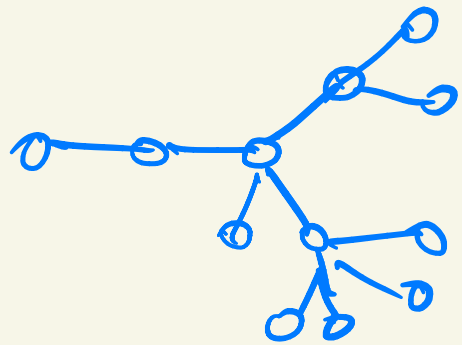


C_5

G $|V| = n$, $|E| = m$

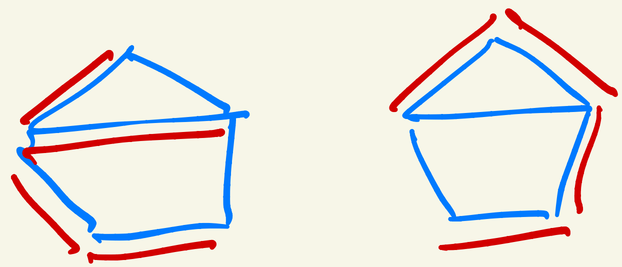
$\boxed{\text{Do}}$ # spanning subgraphs $= 2^m$

tree: connected graph without cycles



Spanning tree

G



Thm (Kirchhoff 1848)

MATRIX-TREE THEOREM

#spanning trees = determinant (matrix associated w graph)

CONTEXT: theory of electrical circuits

CAYLEY'S FORMULA

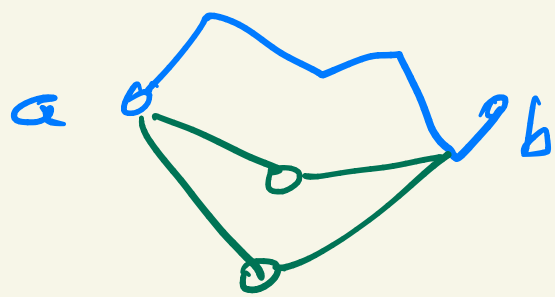
$\#$ spanning trees of K_n — complete graph on n vertices

is n^{n-2}

K_5

$\boxed{\text{HW}}$ PRÜFER code: one of the proofs of CAYLEY's formula

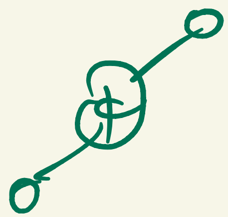
DEF distance between $a, b \in V$



DEF diameter of G

$$\operatorname{diam}(G) = \max_{a,b\in V} \operatorname{dist}(a,b)$$

RANDOM GRAPH



What is the typical diameter?