2022-11-15 | p1

16.55 counting

trivial events: $\# =$ power of 2 $\quad 2^{k+1}$

counting $\{A \subseteq \Omega \mid P(A) = 0\} = Z$

let $B = \{a \in \Omega \mid P(a) = 0\}$ $\quad |B| = k$

then $P(A) = 0 \iff A \subseteq B$ $\quad \therefore \quad Z = \mathcal{P}(B)$

$|Z| = 2^k$

Counting $\{C \subseteq \Omega \mid P(C) = 1\} = W$

Claim $\quad |W| = |Z|$ $\quad$ find bijective proof, i.e., find bijection $f: W \to Z$

$f(C) := \overline{C} = \Omega \setminus C$

18.119 $E(\#\text{Aces})$ in poker hand

size of sample space?

$$\Omega = \binom{C}{5} \qquad C = \text{set of 52 cards}$$

$$\circledast \quad |\Omega| = \binom{52}{5}$$

$X = \#$ Aces $\qquad X_i :=$ indicator of event $\nearrow A_i$

"i-th card is an Ace"

$$X = \sum_{i=1}^{5} X_i$$

$$E(X) = \sum E(X_i) = \sum P(A_i) = \sum_{i=1}^{5} \frac{1}{13} = \frac{5}{13}.$$

X sol'n not consistent with $\circledast$

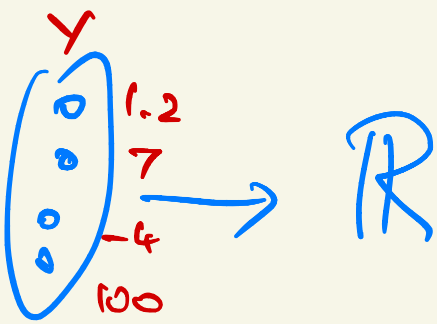
NEED OTHER MODEL

new Sample space: all ordered 5-tuples of distinct cards

$$|\Omega'| = 52\cdot 51\cdot 50\cdot 49\cdot 48 = \binom{52}{5}\cdot \underbrace{5!}_{120}$$

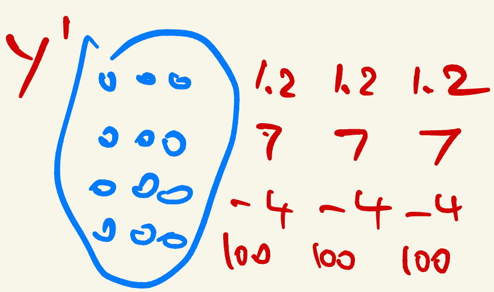
Your proof works.

$\boxed{\text{BLOWING UP SAMPLE SPACE}}$



Ω

$$E(Y) = \frac{1.2 + 7 + (-4) + 100}{4}$$



Ω'

$$E(Y') = \frac{1.2\times 3 + 7\times 3 + (-4)\times 3 + 100\times 3}{4\cdot 3} = E(Y)$$

yields same expected value

<u>2nd</u> Sol'n: uses $\binom{52}{5} = |\Omega|$

LEMMA. Let Q be a card.

$$P(Q \in \text{hand}) = \frac{\binom{51}{4}}{\binom{52}{5}} = \frac{\dfrac{51\cdot 50\cdot 49\cdot 48}{4\cdot 3\cdot 2\cdot 1}}{\dfrac{52\cdot 51\cdot \cdots 48}{5\cdot 4\cdots 1}} = \frac{5}{52}$$

I_i : indicator that Ace_i is in hand $i = 1..4$

$$X = \sum_{i=1}^{4} I_i$$

$$E(X) = \sum E(I_i) = \sum P(\text{Ace}_i \text{ in hand}) = \sum_{i=1}^{4} \frac{5}{52} = \frac{5\cdot 4}{13\cdot 4} = \underline{\underline{\frac{5}{13}}}$$

DIAMETER OF RANDOM GRAPHS

$G = (V, E)$ graph $\quad \text{diam}(G) \overset{\text{DEF}}{=} \max_{u,v \in V} \text{dist}_G(u,v)$

where $\text{dist}_G(u,v) =$ length of shortest $u \cdots v$ path

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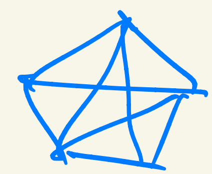
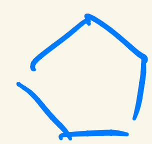
$\text{diam}(K_n) = 1$



K_5

$\text{diam}(C_n) = \left\lfloor \frac{n}{2} \right\rfloor$



$C_5 \to$ diam$=$ 2

$\lfloor x \rfloor = \max\{k \in \mathbb{Z} \mid k \le x\}$ rounded-down value

"floor of x"

Examples: $\lfloor 3.1 \rfloor = 3 \qquad \lfloor -3.1 \rfloor = -4 \qquad \lfloor 3 \rfloor = 3$

$n \to \infty$

Theorem Almost all graphs have diam $= 2$

meaning:

$$p_n = P\left(G \in \mathcal{G}(n, \tfrac{1}{2}) \text{ has diam} = 2\right)$$

uniform ERDŐS-RÉNYI model

Thm $p_n \to 1$

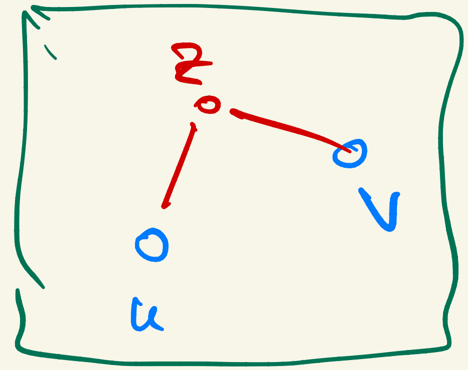
Size of sample space: $2^{\binom{n}{2}}$

Proof: bound $\underbrace{\text{prob. that diam} \neq 2}_{q_n = 1 - p_n}$

NTS: $q_n \to 0$

$$P(\mathrm{diam} = 1) = \frac{1}{2^{\binom{n}{2}}} \longrightarrow 0$$

$$\mathrm{diam}(G) = 1 \iff G = K_n$$



NTS $P(\mathrm{diam} \geq 3) \longrightarrow 0$

$$A := \{G \in \Omega \mid \mathrm{diam}(G) \geq 3\}$$

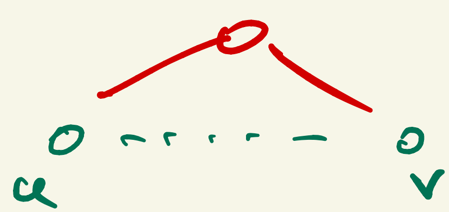
$$B := \{G \in \Omega \mid (\exists u \neq v \in V)(u, v \text{ have no common neighbor})\}$$

$$\text{If } \mathrm{diam}(G) \geq 3 \overset{!}{\Longrightarrow} (\exists u \neq v \in V)(\text{no common neighbor})$$

$$\Downarrow$$

$$(\exists u, v)(\mathrm{dist}(u, v) \geq 3)$$

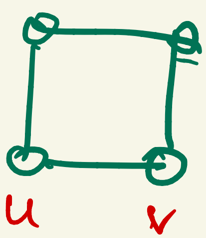
$$\therefore \quad A \subseteq B$$

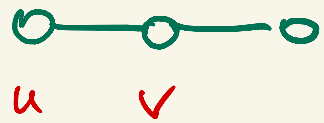


$B \not\Rightarrow A$: give a concrete counterexample

you choose $\underline{\underline{n}}$

$$\left[\begin{array}{l} (\exists u,v \in V)(\text{no common neighbor}) \\ \wedge\ \ \text{diam}(G)=2 \end{array}\right.$$

C_4 



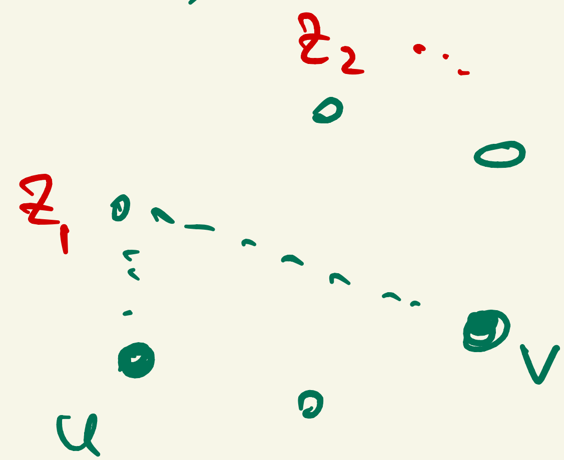
P_3 path of length 2

digression

now back to proof

$$B = \{ G \mid (\exists u \neq v \in V)(\text{no common neighbor}) \}$$

$$C_{u,v} = \{ G \mid u \text{ and } v \text{ have } \text{—"—} \}$$



$u, v, z \in V$ distinct vertices

$$P(z \text{ is a common neighbor of } u,v) = \frac{1}{4}$$

$$P(z \text{ is } \underline{\text{not}} \text{ a common neighbor}) = \frac{3}{4}$$

$$P(\text{none of } z_1, \dots z_k \text{ is a common neighbor of } u,v) = \left(\frac{3}{4}\right)^k$$

b/c events "z_i not —"—" are functions of disjoint blocks of coins flips $\therefore$ <u>independent</u>

$$P(\underbrace{u,v \text{ have no common neighbor}}^{C_{u,v}}) = \left(\frac{3}{4}\right)^{n-2}$$

exponentially decaying

$$B = \bigcup_{\substack{u,v \in V \\ u \neq v}} C_{u,v}$$

$$B = \{G \mid \underbrace{\exists_{\substack{u,v \\ u \neq v}}}\, u,v : \text{no common neighbor}\}$$

quadratic

$$P(B) \leq \sum P(C_{u,v}) = \binom{n}{2} \cdot \left(\frac{3}{4}\right)^{n-2} \to 0$$

union bound

"exponential decay beats polynomial growth"