2022/11/17 P1

$$\sqrt{n^2+1} - n \longrightarrow 0$$

DEF $a_n \sim b_n$ if

$$\frac{a_n}{b_n} \to 1$$

$$a_n \sim a_n \iff a_n \text{ eventually non-zero}$$

$$\underbrace{\sqrt{n^2+1}}_{a} - \underset{b}{n} = \frac{n^2+1-n^2}{\sqrt{n^2+1}+n} =$$

$$\frac{a^2-b^2}{a+b} = a-b$$

$$= \frac{1}{\sqrt{n^2+1}\;+n} \sim \frac{1}{2n}$$

$$\left.\begin{array}{l}\sqrt{n^2+1} \sim n \\ n \sim n\end{array}\right\}$$

$$\frac{\begin{array}{c}a_n \sim b_n \\ c_n \sim d_n\end{array}}{\Rightarrow a_n + c_n \sim b_n + d_n}$$

assuming

$b_n, d_n > 0$

DEF a_n is <u>polynomially bounded</u>

if $(\exists n_0, c)(\forall n \geq n_0)(|a_n| \leq n^c)$

Example $a_n = n^5 + 1000n^2 + 10^6$ $\forall c > 5$

DEF b_n grows at least exponentially

if $(\exists n_0, c>0)(\forall n \geq n_0)(|b_n| \geq e^{n^c})$

ex:

$c=1$ e^n

$c=\frac{1}{2}$ $e^{\sqrt{n}}$

$c=\frac{1}{100}$ $e^{n^{1/100}}$

$\exp(z) = e^z$

$\exp(n^c)$

incl. $c_n = 0$

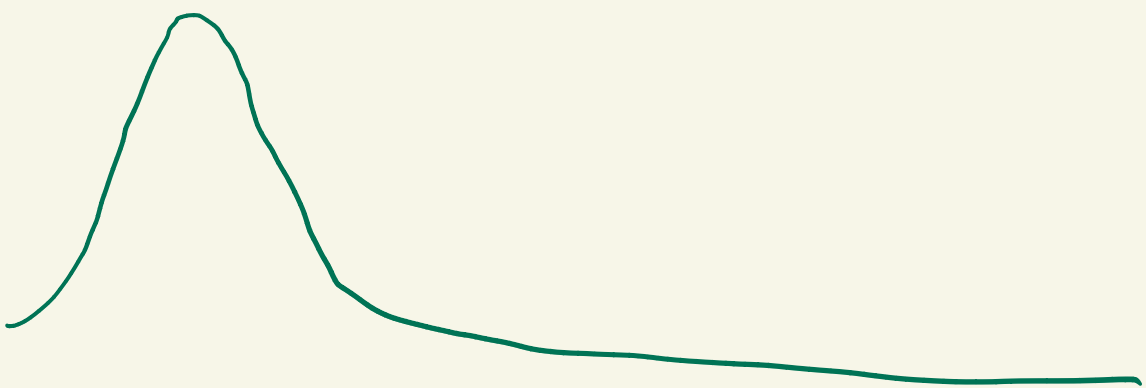
DEF c_n decays (at least) exponentially

if $\left|\frac{1}{c_n}\right|$ growth at least exp.

Thm "exponential decay beats polynomial growth"

THM $(\forall c_1, c_2 > 0)\left(n^{c_1} \cdot \exp\left(n^{-c_2}\right) \to 0 \right)$

<u>Example</u> $n^{100} \cdot \exp\left(n^{-\frac{1}{1000}}\right) \to 0$



LEMMA $\lim_{x\to\infty} \frac{\ln x}{x} = 0$

$\ln x = o(x)$ as $x \to \infty$

L'Hospital's rule: if $f(x) \to \infty$, $g(x) \to \infty$ as $x \to L \in \overline{\mathbb{R}}$

and f, g differentiable

and $\exists \lim \frac{f'(x)}{g'(x)} = K$

then $\exists \lim \frac{f(x)}{g(x)} = K$

$(\ln x)' = \frac{1}{x}$

$x' = 1$

$$\lim_{x\to\infty} \frac{\frac{1}{x}}{1} = \lim_{x\to\infty} \frac{1}{x} = 0$$

✓

COROLLARY

$\frac{x}{e^x} \to 0$ as $x \to \infty$

Pf: Let $y = e^x$ Change of variable

$\therefore$ $x = \ln y$

$$\frac{x}{e^x} = \frac{\ln y}{y} \to 0 \quad \text{as } y \to \infty$$

then. $\frac{x^c}{e^x} \to 0$

$z = \frac{x}{c}$ $\therefore$ $x = cz$

$$\frac{x^c}{e^x} = \left(\frac{x}{e^{x/c}}\right)^c = \left(\frac{cz}{e^z}\right)^c = \left(\cdots\right)^c \to 0$$

$\frac{z}{e^z} \to 0$

as $z \to \infty$ ($\Leftarrow x \to \infty$) $\therefore$ $\frac{cz}{e^z} \to 0$

$\boxed{\text{HW}}$ Use such tricks to prove: $\forall c_1, c_2 > 0$

$$\frac{x^{c_1}}{\exp(x^{c_2})} \longrightarrow 0$$

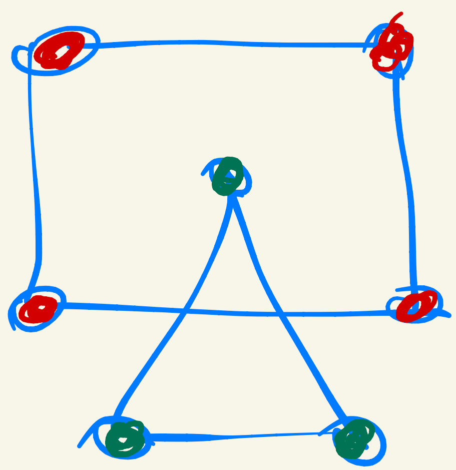
GRAPH THEORY

$G = (V, E)$ $x, y \in V$ vertices

y is accessible from x if $\exists$ $x \cdots y$ path

Observation Accessibility is an equivalence relation on V

subgraphs induced by the blocks of partition: **connected components**



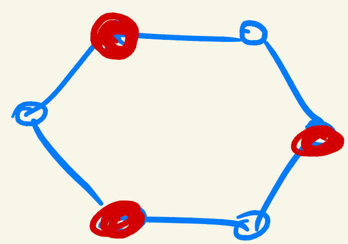
G is connected if there is just 1 component.

$\boxed{HW}$ $(\forall G)(G$ or $\overline{G}$ is connected$)$ $\overline{G}$: complement

DEF G is <u>self-complementary</u> if $G \cong \overline{G}$

- $n=4$ <u>HW.</u> $\exists$ self-compl. graphs with $n=4$, $n=5$ vertices
- $n=5$ <u>HW.</u> If $G \cong \overline{G}$ then $n \equiv 0$ or 1 <u>mod 4</u>

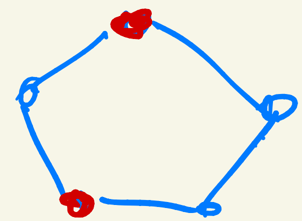
DEF independent set in G is a subset $K \subseteq V$ such that $G[K]$ has no edges



DEF <u>independence number</u>

$$\alpha(G) = \max\{|W| \mid W \text{ indep. set}\}$$

$$\alpha(C_n) = \left\lfloor \frac{n}{2} \right\rfloor$$



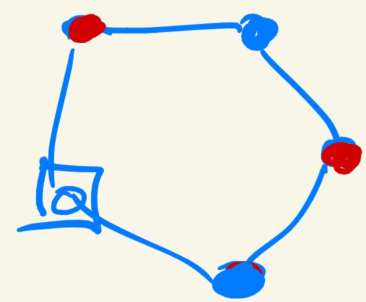
DEF legal coloring of G is a function $f: V \to C$ ← set of colors

s.t. if $f(u)=f(v)$ then $u \not\sim v$

DEF <u>Chromatic number</u>

$\chi(G)$ = min # colors for a legal coloring

\chi



$\boxed{\text{HW}}$ $\alpha(G) \cdot \chi(G) \geq n$

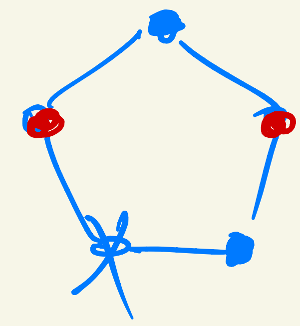
<u>EX</u>

$$\chi(C_n) = \begin{cases} 3 & \text{if } n \text{ is odd} \\ 2 & \text{if } n \text{ is even} \end{cases}$$

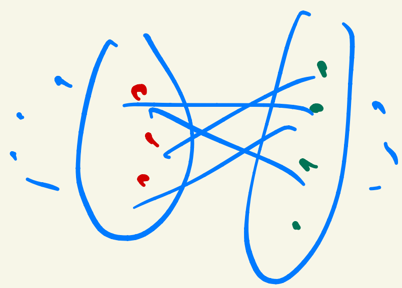
$n \geq 3$

<u>THM</u> G is 2-colorable $(\chi \leq 2)$

$\iff$ G has no odd cycle



DEF 2-colorable : bipartite graph



$$\chi(G) \leq 1 + \deg_{max}$$

$$= \max \{ \deg(u) \mid u \in V \}$$

BONUS If G is triangle-free then $\chi(G) = O(\sqrt{n})$

$$\chi(K_{101}) \geq 101$$

$$\chi(K_n) = n$$

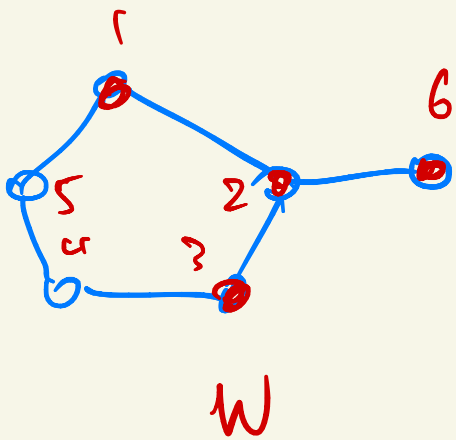
INDUCED SUBGRAPH (by W)

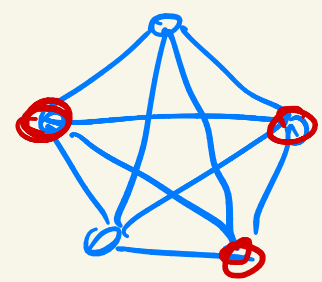
$G = (V, E)$

$W \subseteq V$

$G[W] = \left(W,\ E \cap \binom{W}{2}\right)$

Graph with n vertices has 2^n induced subgraphs





G[W]