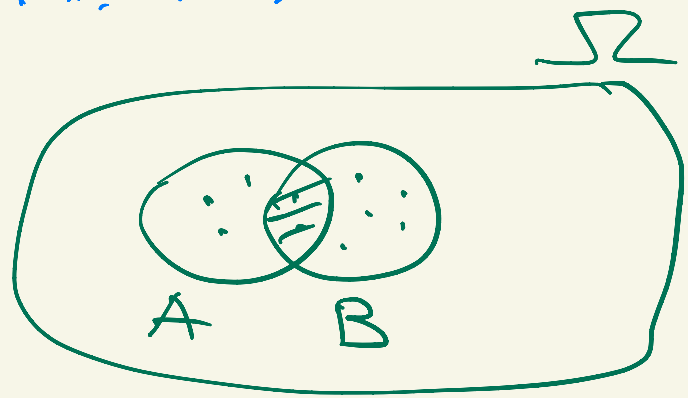
$$P(A\cup B) \le P(A) + P(B)$$

$$P(A\cup B) = P(A) + P(B) - P(A\cap B)$$

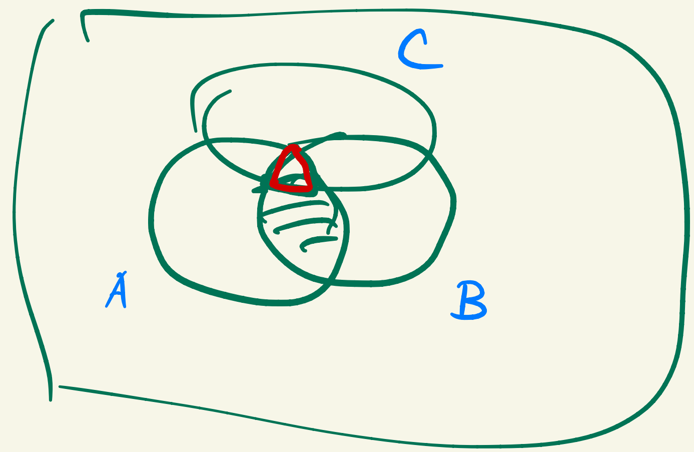


$$P(A\cup B\cup C) =$$

$$P(A)+P(B)+P(C)$$

$$-P(A\cap B)-P(A\cap C)-P(B\cap C)$$

$$+P(A\cap B\cap C)$$



if $x\in A\cap B\cap C$ then

x was counted $3-3+1$ times

p2

$$P(A_1 \cup \cdots \cup A_k) = \underbrace{\sum_{i=1}^{k} P(A_i)}_{S_1} - \underbrace{\sum_{1 \le i < j \le k} P(A_i \cap A_j)}_{S_2} +$$

$$+ \underbrace{\sum_{1 \le i < j < \ell \le k} P(A_i \cap A_j \cap A_\ell)}_{S_3} - + \cdots$$

terms $= \binom{k}{2} = \frac{k(k-1)}{2}$

$\binom{k}{3}$ terms

$$S_t = \sum_{\substack{T \subseteq [k] \\ |T| = t}} P\left(\bigcap_{i \in T} A_i\right)$$

$$\boxed{P(A_1 \cup \cdots \cup A_k) = S_1 - S_2 + S_3 - \cdots}$$

INCLUSION-EXCLUSION formula

INCL-EXCL restated

$$P(\overline{A_1 \cup \dots \cup A_k}) = S_0 - S_1 + S_2 - S_3 + - \dots$$

$$S_0 = 1 = P\Big(\bigcap_{i \in \emptyset} A_i\Big) = P(\Omega)$$

Recall: $\prod_{i=1}^{k} (1+x_i) =$

expansion will have 2^k

$\prod (1-x_i) =$

$(1+x_1)(1+x_2) = 1+x_1+x_2+x_1x_2$

$(1+x_1)(1+x_2)(1+x_3) = 1+x_1+x_2+x_3+x_1x_2+x_1x_3+x_2x_3+x_1x_2x_3$

k	# terms
2	4
3	8

$$\prod_{i=1}^{k}(1+x_i) = \sum_{T \subseteq [k]} \prod_{i \in T} x_i$$

$$\prod_{i=1}^{k}(1-x_i) = \sum_{T \subseteq [k]} \underbrace{(-1)^{|T|}} \prod_{i \in T} x_i$$

INCL-EXCL restated

$$P(\overline{A_1 \cup \cdots \cup A_k}) = \sum_{T \subseteq [k]} (-1)^{|T|} P\left(\bigcap_{i \in T} A_i\right)$$

Let ϑ_i = indicator of A_i

Observation B, C events

$\boxed{\text{Do}}$

$$\vartheta_{\overline{B}} = 1 - \vartheta_B$$

$$\vartheta_{B \cap C} = \vartheta_B \cdot \vartheta_C$$

Proof of INCLUSION-EXCLUSION FORMULA

$$P\left(\overline{A_1 \cup \dots \cup A_k}\right) = E\left(\vartheta_{\overline{A_1 \cup \dots \cup A_k}}\right) = (*)$$

$$\vartheta_{\overline{A_1 \cup \dots \cup A_k}} = \vartheta_{\bar{A}_1 \cap \dots \cap \bar{A}_k} = \prod_{i=1}^{k} \vartheta_{\bar{A}_i} =$$

$$\prod_{i=1}^{k} (1 - \vartheta_{A_i}) = \sum_{T \subseteq [k]} (-1)^{|T|} \prod_{i \in T} \vartheta_{A_i} = \sum_{T \subseteq [k]} (-1)^{|T|} \vartheta_{\bigcap_{i \in T} A_i}$$

$$(*) = \sum_{T \subseteq [k]} (-1)^{|T|} E\left(\vartheta_{\bigcap_{i \in T} A_i}\right) = \sum_{T \subseteq [k]} (-1)^{|T|} P\left(\bigcap_{i \in T} A_i\right)$$

p6

$$P = P(\overline{A_1 \cup \dots \cup A_k}) = S_0 - S_1 + S_2 - + \dots$$

$$P \le S_0$$

$$P \ge S_0 - S_1$$

HW → $$P \le S_0 - S_1 + S_2$$

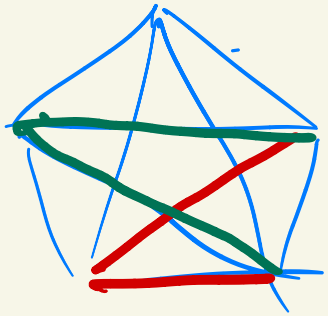
$$P \ge S_0 - S_1 + S_2 - S_3$$

$\vdots$

BONFERRONI's inequalities

Reward : general case

RAMSEY'S THEOREM



avoid $\triangle$

$\triangle$

K_n : $\binom{n}{2}$ moves

if $n \geq 6$ then there is no draw

- $6 \to (3,3)$

coloring edges K_3

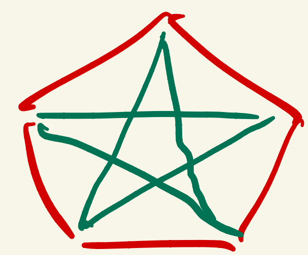
$10 \to (3,4)$

$n \to (k,l)$

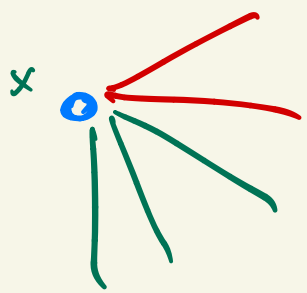
$5 \not\to (3,3)$

C_5

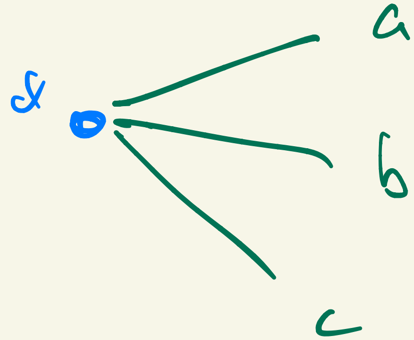
C_5



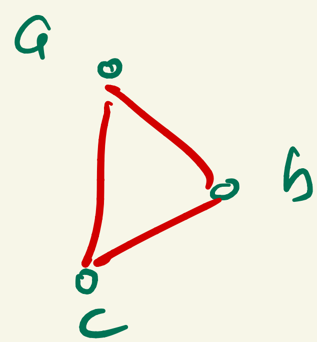
Thm. $6 \to (3,3)$ BABY-RAMSEY



WLOG ≥ 3 neighbors of x are join by green



If $\nexists$ green Δ then



HW $17 \to (3,3,3)$ What does this mean?

THM $(\forall k,l)(\exists n)(n \to (k,l))$

(Ramsey's Thm, very special case) 1928

READ biography of FRANK PLUMPTON RAMSEY

ERDŐS – SZEKERES 1936

$R(k,l)$: smallest n s.t. $n \to (k,l)$

$R(3,3) = 6$ meaning $6 \to (3,3)$

$5 \not\to (3,3)$

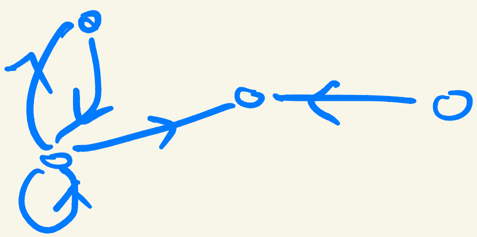
$R(k,l) \le \binom{k+l-2}{k-1}$ $\binom{3+3-2}{3-1} = \binom{4}{2} = 6$

Reward: prove by induction on $\underline{\underline{k+l}}$ $(k,l \ge 2)$

HW BASE CASE : $k=2$ OR $l=2$

DIRECTED GRAPHS (digraphs)

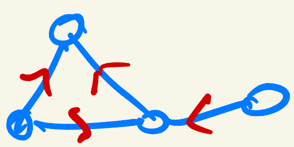
$G = (V, E)$ $\quad$ $E \subseteq V \times V$ relation on V



(self-loop)

If no self-loop: E is an irreflexive rel.

ORIENTATION OF A GRAPH

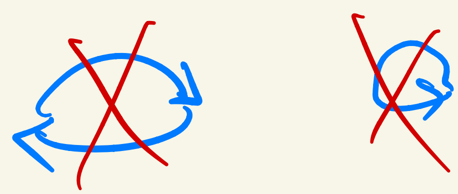


n vertices

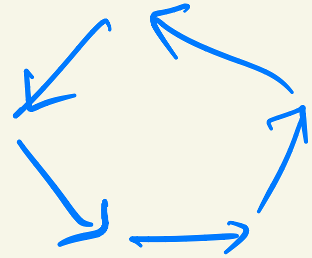
m edges

$\boxed{\#\text{ orientations: } 2^m}$

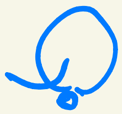
ORIENTED GRAPH: anti symm. digraph

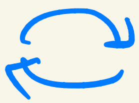


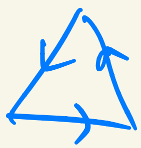
directed cycle



$\vec{C}_5$

 $\vec{C}_1$

 $\vec{C}_2$

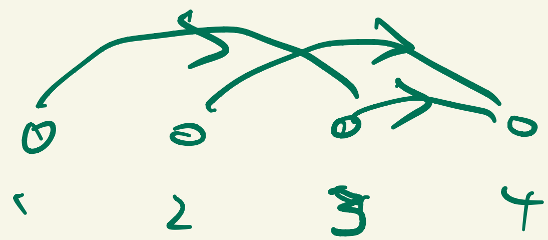
 $\vec{C}_3$

⋮

DAG directed acyclic graph:
digraph without cycles ∴ oriented

---

topological sort of a digraph is
an ordering of V s.t.
$(\forall (i,j) \in E)(\, i < j \,)$



Let G be a digraph.

Thm G is a DAG $\iff$ G has a top. sort

straightforward direction: $\Leftarrow$

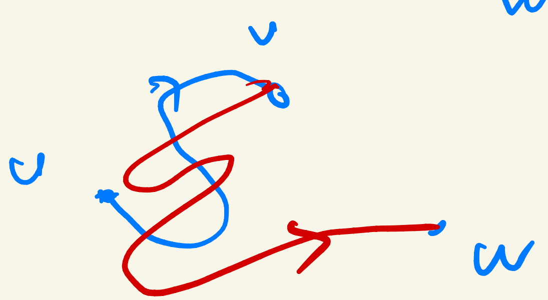
directed path $\circ \to \circ \to \circ \to \circ \to \circ$ $\vec{P}_5$

vertex $\underline{v}$ accessible from $\underline{u}$ if $\exists\ u \to \dots \to v$ dir. path

reflexive: $\underline{u}$ is accessible from $\underline{u}$ by $\vec{P}_1$ (length zero)

$\boxed{\text{HW}}$ transitive:

if v is acc. from u
w " " v $\Big\}$ then w is accessible from u



mutual accessibility: $\underline{u}$ accessible from $\underline{v}$
and v acc from u

→ equivalence relation on V

Eq. classes: strong components