12-01-2022 | P 1

poker hand $|\Omega| = \binom{52}{5}$

$\Omega = \binom{[52]}{5}$

$X := \#$ Aces
$Y := \#$ Spades } in hand

NOT independent:
$P(X=4, Y=0) = 0$
$\neq P(X=4) \cdot P(Y=0)$

Prove X, Y: uncorrelated

$\Omega' :=$ ordered hands $\quad |\Omega'| = |\Omega| \cdot 5!$

X_i : indicator that i^{th} card is Ace
Y_i : " " " is Spades

$X = X_1 + \dots + X_5$
$Y = Y_1 + \dots + Y_5$

$E(X) = \frac{5}{13}$

$E(Y) = \frac{5}{4}$

$E(XY) = \sum_i \sum_j E(X_i Y_j) \rightarrow \frac{5 \cdot 5}{52} = E(X) \cdot E(Y)$ ✓

$E(X_i Y_j) = P(i^{th}: \text{Ace} \wedge j^{th}: \text{Spades}) = \begin{cases} \frac{1}{52} & \text{if } i=j \\ \frac{1}{52} & \text{again} \end{cases}$ $\leftarrow \boxed{\text{DO}}$

RAMSEY

$n \to (k,\ell)$ means:

no matter how we color $E(K_n) = R \,\dot\cup\, B$,

either $\underbrace{([n], R)}_{\text{red graph}} \supseteq K_k$

or $\underbrace{([n], B)}_{\text{blue graph}} \supseteq K_\ell$

Ramsey number $R(k,\ell) := \min\{n \mid n \to (k,\ell)\}$

$6 \to (3,3)$ $\qquad$ $\underline{\underline{6}}$ is tight: $5 \not\to (3,3)$

$\underline{\text{i.e.}}$ $R(3,3) = 6$

ERDŐS – SZEKERES

$$\binom{k+\ell-2}{k-1} \to (k,\ell)$$

$$\binom{3+3-2}{3-1} = \binom{4}{2} = 6 \to (3,3)$$

$\underline{\underline{\text{i.e.}}}$ $R(k,\ell) \le \binom{k+\ell-2}{k-1}$

$R(k,\ell) = R(\ell,k)$

$$R(3,4) \leq \binom{3+4-2}{3-1} = \binom{5}{2} = 10$$

Bonus $R(3,4) \leq 9$ i.e. $9 \to (3,4)$

Diagonal Ramsey numbers $R(k) = R(k,k)$

$k = \ell$

$R(3) = 6$

$R(4) = 18$

$43 \leq R(5) \leq 48$

$102 \leq R(6) \leq 161$

$205 \leq R(7) \leq 497$

$282 \leq R(8) \leq 1532$

$$4^k > \binom{2k}{k} \to (k+1, k+1)$$

$\boxed{\text{DO}}$ $k \geq 6 \Rightarrow$

$$4^{k-1} \geq \binom{2k}{k}$$

$$\sum_{i=0}^{2k} \binom{2k}{i} = 2^{2k} = 4^k$$

$$n := 4^k \to (k+1, k+1)$$

$$n \to \left(1 + \tfrac{1}{2}\log_2 n,\ 1 + \tfrac{1}{2}\log_2 n\right)$$

$R(k) < 4^k = 2^{2k}$

$R(k) > ?$

HW

$k^2 \not\to (k+1, k+1)$

explicit, simply stated coloring

THEOREM (ERDŐS 1950)

$R(k) > 2^{\frac{k-1}{2}}$

$2^{k/2} \not\to (k+1, k+1)$

LEMMA Almost all graphs $G(n, \frac{1}{2})$ uniform E-R model

$\alpha(G) < 1 + 2\log_2 n$

$\therefore$ a.a. $\omega(G) < 1 + 2\log_2 n$

$\therefore (\exists G)(\alpha(G), \omega(G) < 1 + 2\log_2 n)$

$\therefore n \not\to (1 + 2\log_2 n, 1 + 2\log_2 n)$

PROBABILISTIC METHOD:

proof of existence w/o explicit construction

first explicit construction that beats $\sqrt{n}$ bound

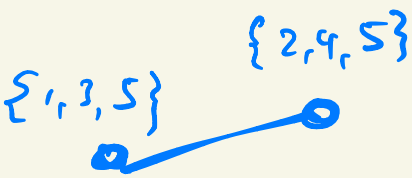
Z. NAGY (1973) $n=\binom{k}{3} \not\to (k+1, k+1)$

$$n \not\to (c\sqrt[3]{n}, c\sqrt[3]{n})$$

Construction: $V=\binom{[k]}{3}$

adjacency: $A, B \subseteq [k]$ $|A|=|B|=3$

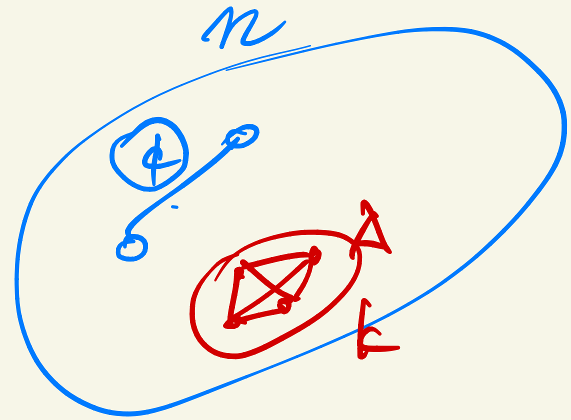
$$A \sim B \text{ if } |A \cap B| = 1$$



Claim: $\alpha, \omega \le k$

PROOF of LEMMA

unif. E-R random graph n vertices



$$A \subseteq [n], \quad |A| = k$$

$$P(A \text{ is indep}) = \frac{1}{2^{\binom{k}{2}}}$$

union bound

$$P(\exists A, |A| = k, \text{indep}) \le \binom{n}{k} \cdot \frac{1}{2^{\binom{k}{2}}}$$

$$\shortparallel$$

$$P(\alpha(G) \ge k) \le \quad \text{—}\, \shortparallel \,\text{—} \quad \le \frac{n^k}{k!} \cdot \frac{1}{2^{\binom{k}{2}}}$$

$\therefore$ if $\underline{\dfrac{n^k}{2^{\binom{k}{2}}}} \le 1$ then $P(\alpha(G) \ge k) \le \dfrac{1}{k!}$

$$\left(\frac{n}{2^{\frac{k-1}{2}}}\right)^k \overset{?}{\le} 1 \iff \frac{n}{2^{\frac{k-1}{2}}} \le 1 \iff n \le 2^{\frac{k-1}{2}}$$

$$\iff \log_2 n \le \frac{k-1}{2} \iff k \ge 1 + 2\log_2 n$$

$n \to (1+\frac{1}{2}\log_2 n, \text{ same})$ E–Sz 1936

$n \not\to (1+2\log_2 n, \text{ same})$ Erdős 1950

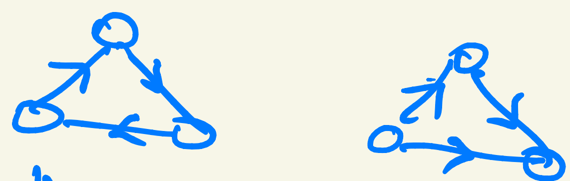
OPEN $n \overset{?}{\to} \left(\left(\frac{1}{2}+\varepsilon\right)\log_2 n, \text{ same}\right)$

$n \overset{?}{\not\to} \left((2-\varepsilon)\log_2 n, \text{ same}\right)$

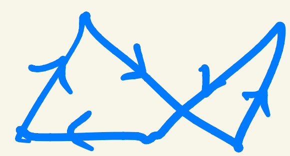
PAUL ERDŐS (1913–1996) & FRIENDS: PAUL TURÁN, GEORGE SZEKERES, ESZTER KLEIN

DEF tournament: orientation of K_n

EX.



1.) All tournaments have a Hamilton path



2.) All strongly conn. tourn. have Hamilton cycle

H-cycle