Predicate over a set A

is a function $A \to \{0,1\}$

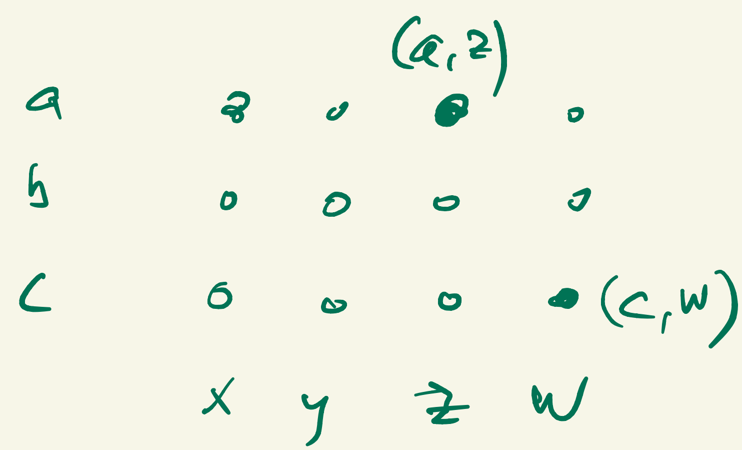
$|A| = n \implies$ # predicates 2^n

Cartesian product $A \times B = \{(a,b) \mid a \in A, b \in B\}$

$|A \times B| = |A| \cdot |B|$

Relation from A to B is a predicate on $A \times B$

domain ↓ codomain ↓



Relation **on A** ← "homogeneous"

a predicate on $A \times A$

HW If $|A| = m$ and $|B| = k$, what is the number of relations from A to B? (closed-form expression)

Examples: "$\leq$" $\begin{cases} (3,5) \to \text{YES} \\ (5,3) \to \text{NO} \end{cases}$

fix m: congruence mod m

divisibility $a \mid b$

$R : A \times A \to \{T, F\}$

DEF R is reflexive if $(\forall a \in A)(R(a,a) = T)$

R is symmetric if $(\forall a, b)$(if $R(a,b) = T$ then $R(b,a) = T$)

R is transitive if $(\forall a, b, c)$

$$\left.\begin{matrix} R(a,b) = T \\ R(b,c) = T \end{matrix}\right\} \Longrightarrow R(a,c) = T$$

$\leq$

$\equiv$ mod m

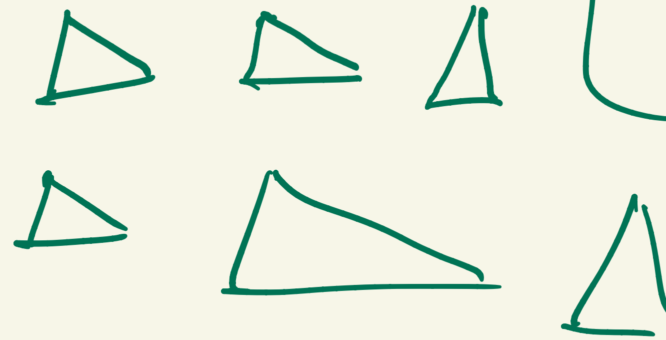
$\mid$

not $<$

DEF R is an equivalence relation if R has each of these 3 properties

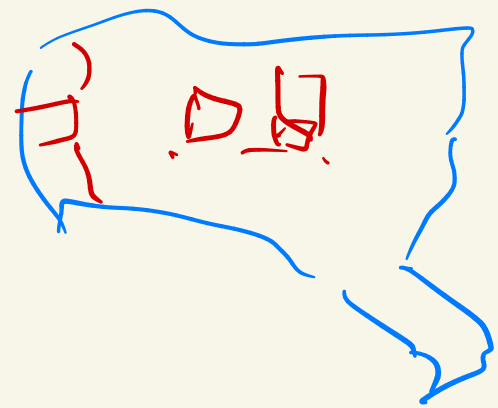
triangles: congruence

similarity



"has the same area"

taxpayers of US



Partition of A:

$$\Pi = \{B_1, \dots, B_k\}$$

$B_i \neq \emptyset$ nonempty

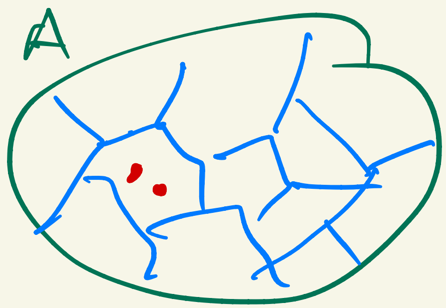
$i \neq j \Rightarrow B_i \cap B_j = \emptyset$ pairwise disjoint

$\bigcup B_i = A$

Given a partition $\Pi = \{B_1, \dots, B_k\}$

take the relation R_Π defined as

$$R_\Pi(x,y) = T \quad \text{if} \quad \underbrace{(\exists i)(x,y \in B_i)}_{x,y \text{ are in the same block}}$$



$\Pi \to R_\Pi$

XC

FUNDAMENTAL THEOREM OF EQUIVALENCE RELATIONS

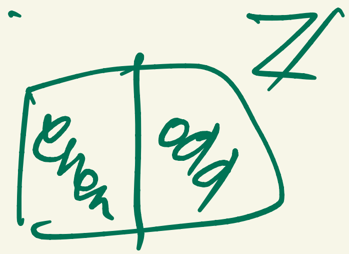
$\forall$ equivalence relation T
$\exists$ partition Π
s.t. $T = R_\Pi$

The blocks of Π are called the equivalence classes of T

TERMINOLOGY The equivalence classes of congruence mod m are called RESIDUE CLASSES modulo m

Residue classes mod 2: even numbers, odd – " –

$x \equiv y \pmod 2$ means $2 \mid x-y$



NOTATION

$A \subseteq \mathbb{Z}$

$c \in \mathbb{Z}$

$c \cdot A = \{ c \cdot a \mid a \in A \}$

$A + c = \{ a + c \mid a \in A \}$

EX $3 \cdot \{-5, 2, 7\} = \{-15, 6, 21\}$

$\{-5, 2, 7\} + 10 = \{5, 12, 17\}$

$\{\text{even numbers}\} = 2 \cdot \mathbb{Z}$

$\{\text{odd numbers}\} = 2 \cdot \mathbb{Z} + 1$

$$2\mathbb{Z}+8 = 2\mathbb{Z}$$

$$2\mathbb{Z}+2023 = 2\mathbb{Z}+1$$

...	...	...	...	-6
-5	-4	-3	-2	-1
0	1	2	3	4
5	6	7	8	9
10	11	12	13	14
...	...	...	...	...
$5\mathbb{Z}$	$5\mathbb{Z}+1$	$5\mathbb{Z}+2$	$5\mathbb{Z}+3$	$5\mathbb{Z}+4$

$\cancel{5}$

$-1 \equiv 4 \mod 5$

residue classes mod m is $\begin{cases} |m| & \text{if } m \neq 0 \\ \infty & \text{if } m = 0 \end{cases}$

$m \in \mathbb{Z}$

$$x \equiv y \mod 0 \iff 0 \mid x-y \iff x-y=0 \iff x=y$$

residue classes modulo 0 are singletons:

$\ldots, \{-3\}, \{-2\}, \{-1\}, \{0\}, \{1\}, \{2\}, \{3\}, \ldots$

mod 7

$[a] := 7\mathbb{Z} + a = \{x \mid x \equiv a \bmod 7\}$

$[b] = 7\mathbb{Z} + b$

mod 7

$[a] + [b] := [a+b]$

changing the representative of the residue class does not change the "vote"

$[-5] + [18] = [13]$

$[16] + [4] = [20] = [13]$

-7	-6	-5	-4	-3	-2	-1
0	1	2	3	4	5	6
7	8	9	10	11	12	13
14	15	16	17	18	19	20
21	22	23	24	25	26	27

$\boxed{\text{HW}}$ If $\left.\begin{array}{l} a \equiv x \pmod{m} \\ b \equiv y \pmod{m} \end{array}\right\} \Longrightarrow a+b \equiv x+y \ (m)$

$\boxed{\text{HW}}$ ——— " ——— $\Longrightarrow a \cdot b \equiv x \cdot y \ (m)$

$\boxed{\text{Do}}$ ——— " ——— $\Longrightarrow a-b \equiv x-y \ (m)$