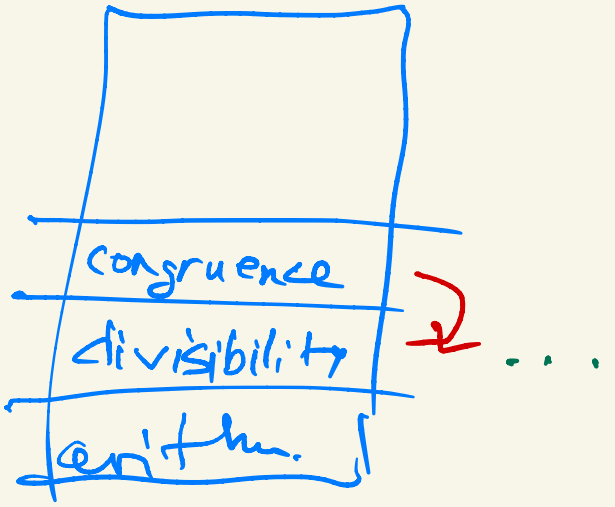
PROBLEM SESSION | 2023-10-13

5.73 $\left.\begin{array}{l} x \equiv y \ (m) \\ y \equiv z \ (m) \end{array}\right\} \Rightarrow x \equiv z \ (m)$

Assm. $\begin{cases} m \mid x-y \\ m \mid y-z \end{cases}$ $\exists$ $m \mid x-z$



Lemma

$\begin{cases} a \mid b \\ a \mid c \end{cases}$
$\overline{a \mid b+c}$ ✓

$(\exists k)(x-y=km)$ not needed

$a := m$

$b := x-y$

$c := y-z$

$\therefore \quad m \mid (x-y)+(y-z) = x-z$ ✓

ASSN x odd

DC $x^2 \equiv 1 \ (8)$ i.e. $8 \mid x^2-1 = (x-1)(x+1)$

obs $\left.\begin{matrix} 2 \mid x+1 \\ 2 \mid x-1 \end{matrix}\right\}$ b/c x odd

NTS $2 \mid \frac{x-1}{2} \cdot \frac{x+1}{2}$ (k, $k+1$)

↳ two consecutive integers
⇒ one of them is even ✓

Other proof: $x = 2k+1$

$x^2-1 = (2k+1)^2-1 = 4k^2+4k = 4 \cdot \underbrace{k(k+1)}_{\text{even}}$ ✓

p prime, $p \geq 5 \Rightarrow p \equiv \pm 1 \ (6)$

$\uparrow$

1 OR -1

$(\forall x)(\exists k)(0 \leq k \leq 5 \wedge x \equiv k \ (6))$

Case: $p \equiv 0 \ (6)$ i.e. $6 \mid p$ $<^{2 \mid p}_{3 \mid p}$ $\rightarrow\leftarrow$

$p \equiv 1 \ (6)$ ✓

$p \equiv 2 \ (6)$ i.e. $2 \mid 6 \mid p-2$

$\therefore 2 \mid p-2$

$\therefore 2 \mid (p-2)+2 = p$

but $\mathrm{Dist}^+(p) = \{1, p\}$ $\Rightarrow p=2$ $\rightarrow\leftarrow p \geq 5$

$p \equiv 3 \ (6)$ $3 \mid 6 \mid p-3$

$\therefore 3 \mid p-3$

$\therefore 3 \mid (p-3)+3 = p$ $\Rightarrow p=3$ $\rightarrow\leftarrow p \geq 5$

-6	-5	-4	-3	-2	-1
0	1	2	3	4	5
6	7	8	9	10	11
12		⋮			
		x			

residue classes mod 6

$p \equiv 4 \ (6)$ i.e. $2 \mid 6 \mid p-4$

$\therefore \ 2 \mid p-4$

$2 \mid 4$

$2 \mid (p-4)+4 = p \implies p = 2 \quad \rightarrow\leftarrow \quad p \geq 5$

$p \equiv 5 \ (6)$

$5 \equiv -1 \ (6)$

$6 \mid 5-(-1) = 6$ ✓

$p \equiv -1 \ (6)$ ✓

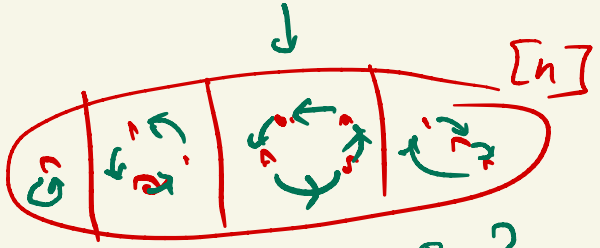
Bell numbers

4.36 $B(n) \le n!$ injective proof: Find $f: P_n \to Q_n$ injection

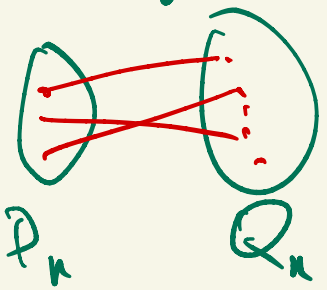
P_n = set of partitions of $[n] := \{1, 2, \dots, n\}$ $\quad |P_n| \overset{\text{def}}{=} B(n)$

Q_n = set of permutations of $[n]$ $\quad |Q_n| = n!$

DEF f is a permutation of the set A if f is an $f: A \to A$ bijection



$\Pi = \{ B_1, B_2, \dots, B_k \}$



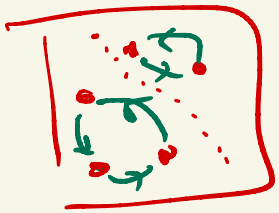
Let $f(\Pi)$ be the permutation that cyclically permutes each block (in increasing order, + last one $\mapsto$ first)

$B_j := \{5, 7, 10, 11\}$

$\to 5 \to 7 \to 10 \to 11 \to$

$(5, 7, 10, 11)$

$(7, 10, 11, 5)$



D.C.

$$\binom{n}{k} \geq \left(\frac{n}{k}\right)^k$$

ASSN

$1 \leq k \leq n$

A ⇑ B

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!} = \frac{n}{k}\cdot\frac{n-1}{k-1}\cdot\frac{n-2}{k-2}\cdots\frac{n-k+1}{1}$$

$$\left(\frac{n}{k}\right)^k = \frac{n}{k}\cdot\frac{n}{k}\cdot\ldots\cdot\frac{n}{k}$$

(k terms)

Lemme ⇒ Thm ✓

termwise comparison

Lemma

for $0 \leq i \leq k-1$ $\quad \frac{n-i}{k-i} \geq \frac{n}{k}$

$$kn - ki = k(n-i) \overset{?}{\geq} n(k-i) = nk - ni$$

$$-ki \overset{?}{\geq} -ni$$

$$ki \overset{?}{\leq} ni \Longleftarrow k \leq n \ ✓$$

What matters is ⇑

5.88 universe is $\mathbb{Z}$

DEF x has the prime property if

$$(\forall a,b)(x \mid ab \implies x \mid a \vee x \mid b)$$

$\therefore$ x does not have the prime property if and only if

$$(\exists a,b)(x \mid ab \wedge x \nmid a \wedge x \nmid b) \quad \longleftarrow$$

$\rightarrow$ 6 does not have the prime property: $a := 3$, $b := 2$ | 3 | 15 ; 4 | 20 ✓

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① If x is a $\pm$prime number
then x has the prime property + EUCLID'S LEMMA

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② If x is a $\pm$ composite number
is: $(\exists a,b)(x = ab,\ a,b \geq 2)$
then x does not have the prime property

WHAT ABOUT $0, \pm 1$ $\quad \pm 1 \mid a,b$ YES

i.e. $A \implies B$

$\neg B \implies \neg A$

Contrapositive

Q T/F $(\forall a,b)(0 \mid ab \Rightarrow 0 \mid a \vee 0 \mid b)$

$\Updownarrow$

$(\forall a,b) \quad ab = 0 \overset{?}{\Rightarrow} a = 0 \vee b = 0$

$\therefore$ 0 has the prime property

$Div(a) \subseteq Div(b) \iff a \mid b$

DEF $Div(a) = \{x : x \mid a\}$

$Div(6) = \{\pm 1, \pm 2, \pm 3, \pm 6\}$

$\Longrightarrow$

$b \mid c \quad a \mid a$

$\downarrow$

$b \mid c \quad a \in Div(a) \subseteq Div(b)$

$\therefore a \in Div(b)$

$\therefore a \mid b$

$\Longleftarrow$ TO