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FACT generalization of Euclid's Lemma

If $a \mid bc$ and $\gcd(a,b)=1$ then $a \mid c$

EX $\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$ $1 \leq k \leq n$

EX $n+1 \mid \binom{2n}{n}$

Solution to $\boxed{DO}$ $\quad n! \geq \left(\frac{n}{e}\right)^n \quad n \geq 0$

Pf. $\quad e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \qquad 0! = 1$

$\therefore$ $\forall x \geq 0$, $\forall k \geq 0$ we have $e^x \geq \frac{x^k}{k!}$

$x \in \mathbb{R}$
$k \in \mathbb{Z}$

$k := n$
$x := n$

NTS $\quad n! \geq \left(\frac{n}{e}\right)^n$

i.e. $\quad e^n \geq \frac{n^n}{n!}$

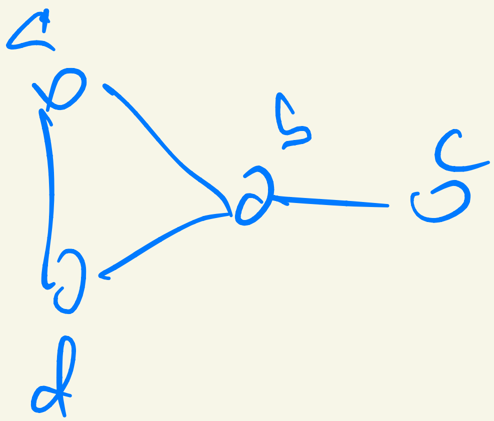
GRAPH

$G = (V, E)$

$|V| = n$
$|E| = m$

V set (of vertices)

E set of unordered pairs of distinct elements of V (set of edges)



$V = \{a, b, c, d\}$

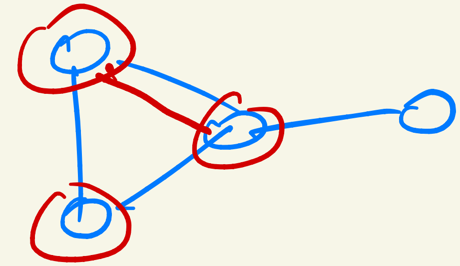
$E = \{ab, bc, bd, ad\}$

where shorthand $\{a, b\} = ab$

$= \{\{a, b\}, \{b, c\}, \{b, d\}, \{a, d\}\}$

$$E \subseteq \binom{V}{2}$$

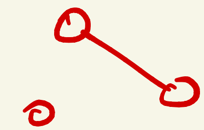
$$\left|\binom{V}{2}\right| = \binom{n}{2} \quad \therefore \quad m \le \binom{n}{2}$$



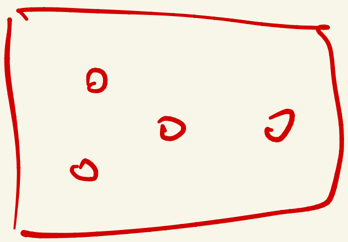
SUBGRAPH

$G = (V, E)$

$H = (W, F)$



$H \subseteq G$ H is a subgraph of G if



$W \subseteq V$ and $F \subseteq E$

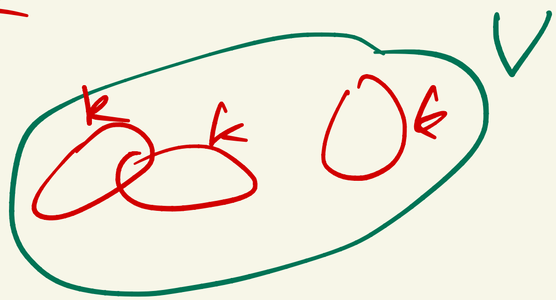
Note: $F \subseteq \binom{W}{2}$ $\therefore$ $F \subseteq E \cap \binom{W}{2}$

Notation V - set

$|V| = n$ $k \geq 0$ integer

$$\binom{V}{k} = \{B \mid B \subseteq V,\ |B| = k\}$$

set of all k-subsets of V



$$\left|\binom{V}{k}\right| = \binom{n}{k}$$

$m = \binom{n}{2} \iff G$ is a clique (a complete graph)

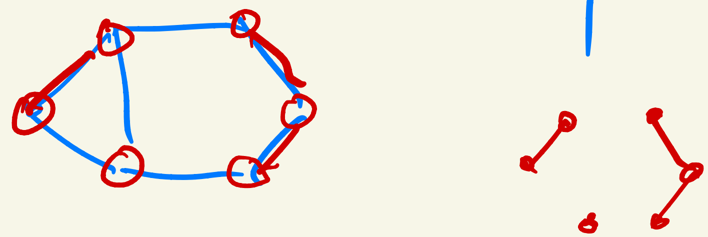
$$G \cong K_n$$

Obs Every graph is a subgraph of a clique

DEF H is a spanning subgraph of G if $W = V$

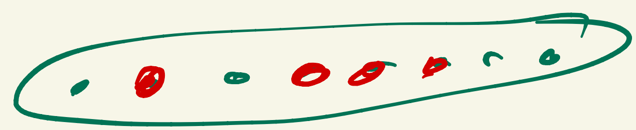
$G = (V, E)$

$H = (W, F)$



\# spanning subgraphs of G

$$2^m$$



0 1 0 1 1 1 0 0

H is an <u>induced</u> subgraph of G

$G = (V, E)$

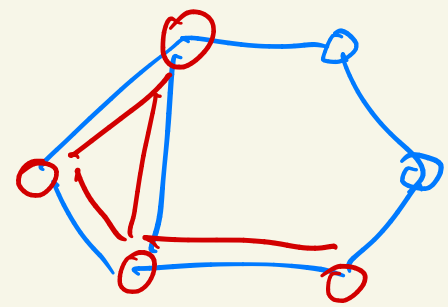
$H = (W, F)$

if $W \subseteq V$

$F = E \cap \binom{W}{2}$

<u>Notation</u>: $H = G[W]$

SUBGRAPH OF G
INDUCED BY W



[ ] – BRACKET

( ) – PARENTHESES

{ } – BRACES

induced subgraphs

2^n

ACCESSIBILITY

$y \in V$ is accessible from $x \in V$ if

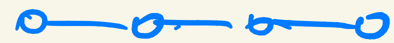
$\exists$ $x - \dots - y$ path is G

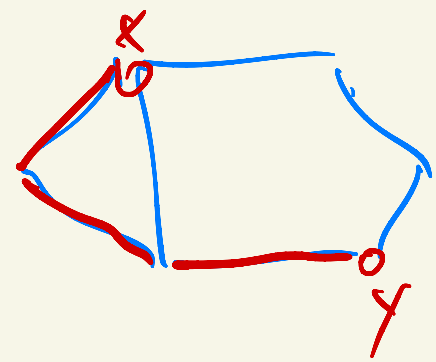
accessibility is a relation on V

this relation is an equivalence relation

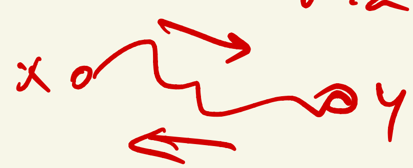
path in G is a subgraph isomorphic to a path

P_4





- reflexive: x acc. from x via P_1
- symmetric:

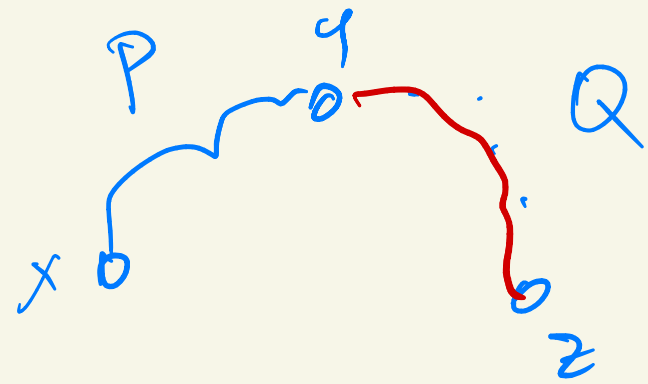


- transitive

z accessible from y, y accessible from x $\Rightarrow$ z accessible from x

transitivity of accessibility

L8



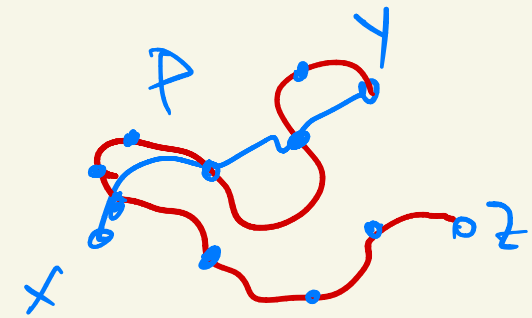
Assn:

$\exists P$, $x \cdots y$ path

$\exists Q$: $y \cdots z$ path

DC:

$\exists R$: $x \cdots z$ path



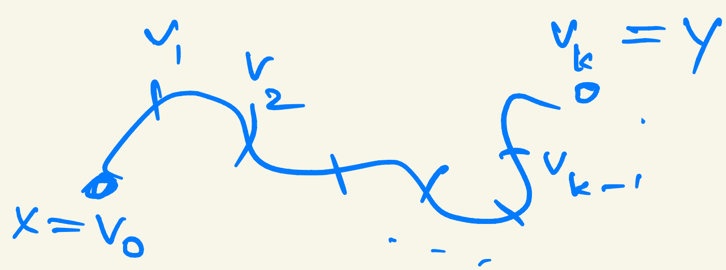
DEF $x \cdots y$ **WALK** of length k

is a string of vertices

$$x = v_0, v_1, \ldots, v_k = y$$

$$\text{s.t. } (\forall i)(v_{i-1} \sim v_i)$$

adjacent



DIRECTED

LEMMA If $\exists x \cdots y$ WALK then $\exists x \cdots y$ PATH

DO

in fact: a shortest walk never repeats a vertex

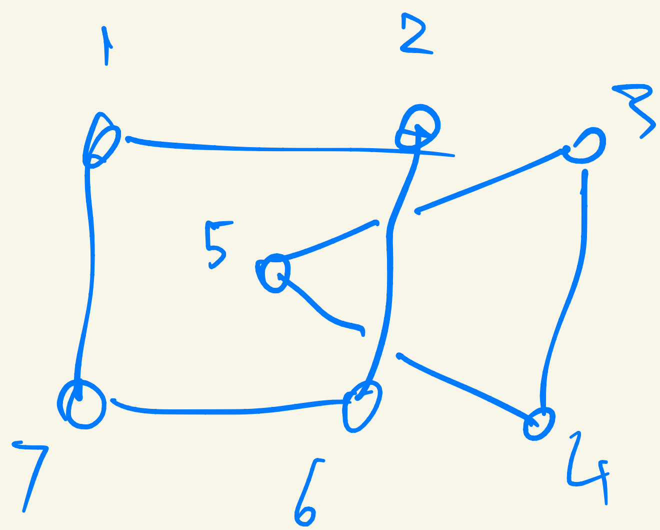
LEMMA (trivial) If $v_0 \cdots v_k = w_0$ and $w_0 \cdots w_\ell$ are walks then their CONCATENATION

$v_0 \cdots v_k = w_0 \cdots w_\ell$ is also a walk

length: $k+\ell$

DEF subgraphs induced by the equivalence classes of "accessibility" are called **connected components** of G

x, y ∈ V are in the same connected component exactly if y is accessible from x



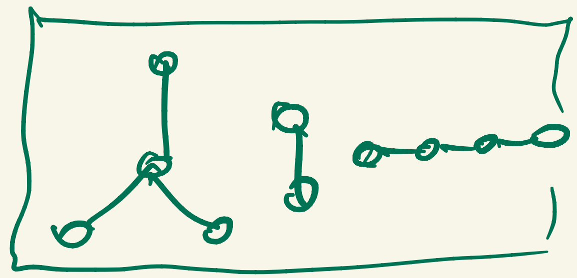
$\{1, 2, 7, 6\}$ $\{3, 4, 5\}$
partition of V

Conn. comp's are the subgraphs induced by these blocks

G is <u>connected</u> if there is just 1 connected component,

<u>i.e.</u> $(\forall x,y \in V)(\exists\, x \cdots y \text{ path})$

<u>DEF</u> <u>Forest</u>: graph without cycles



<u>Tree</u>: connected forest: connected cycle-free graph

<u>Obs.</u> Every connected component of a forest is a tree