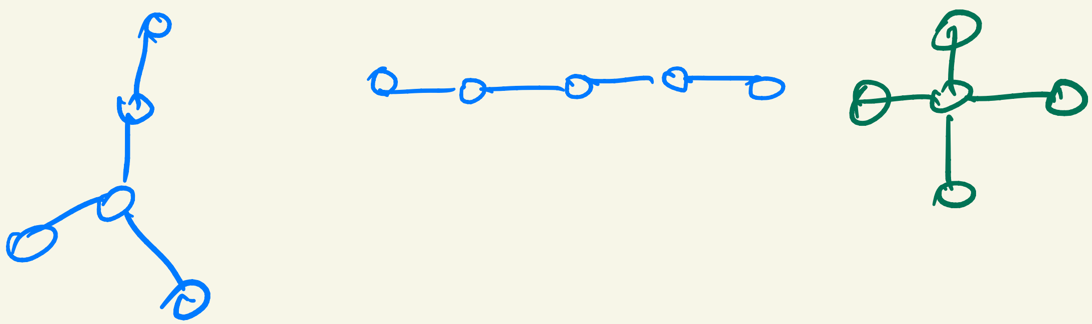
2023-10-19 | 1

TREE

Connected, cycle-free graph



$n = |V|$ order of graph

$m = |E|$ size of graph

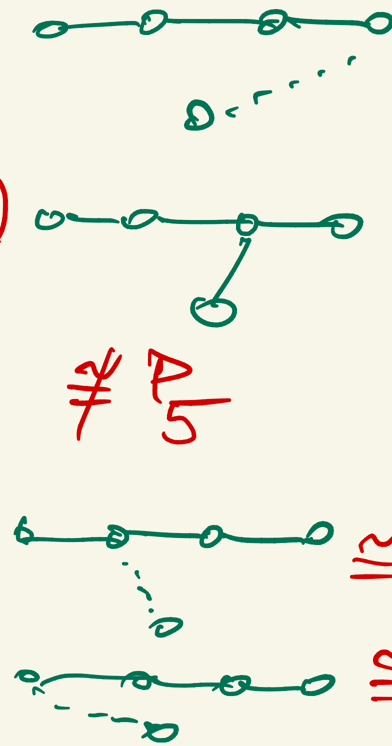
Thm Tree of order $n \geq 1$ has size $n-1$

Sort trees of order 5 by
$\ell :=$ length of longest path

$\ell = 4 \longrightarrow T \cong P_5$

1

$\ell = 3$



P_5 X

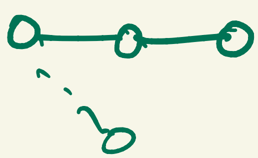
2 new

$\not\cong P_5$

$\cong$ 2 X

$\cong P_5$

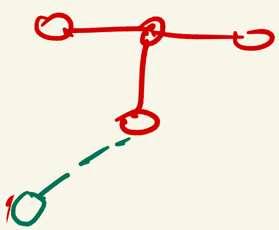
$\ell = 2$



X

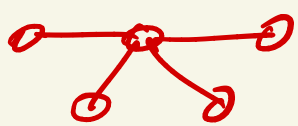
$\ell \geq 3$

b/c P_5 has no vertex of deg 3 $\longrightarrow$



X

$\longrightarrow$ only possibility



3

$\ell = 1$ X

$G = (V, E)$

$V = V(G)$

$E = E(G)$

Subgraph $H = (W, F)$

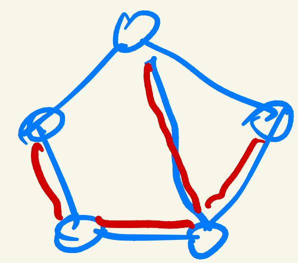
$W = V(H)$

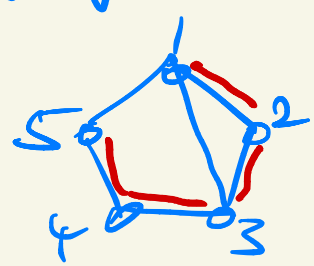
$F = E(H)$

DEF $H \subseteq G$ subgraph if $W \subseteq V$ and $F \subseteq E$

H is a spanning subgraph if $W = V$

DEF H is a spanning tree of G if $H \subseteq G$ and H is spanning, i.e. $W = V$

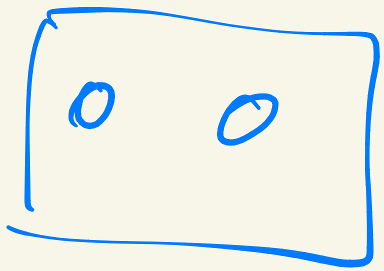


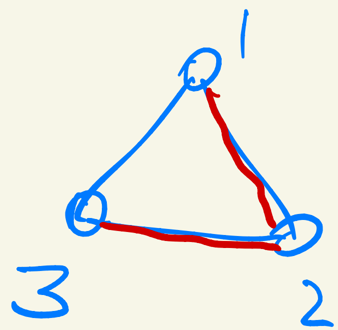


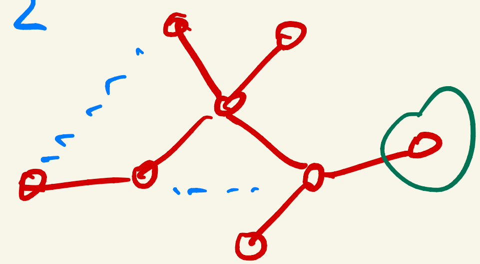
Kirchhoff : determinant formula
1848
for # spanning trees of any graph

CAYLEY'S FORMULA

spanning trees of K_n $\quad n^{n-2}$





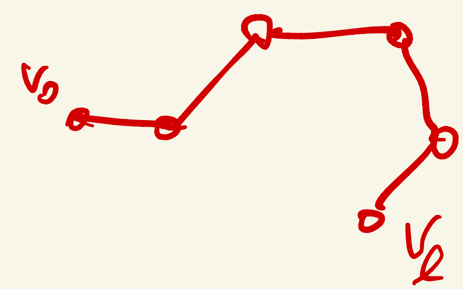


G has a spanning tree $\iff$ G is connected

Pf $\Longrightarrow$ obvious

$\Longleftarrow$ $\boxed{\text{DO}}$

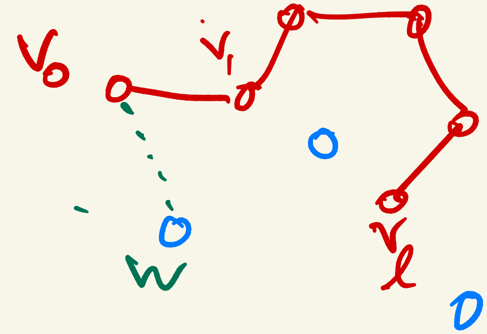
Thm Every tree of order ≥ 2 has a vertex of deg 1 (pendant vertex)

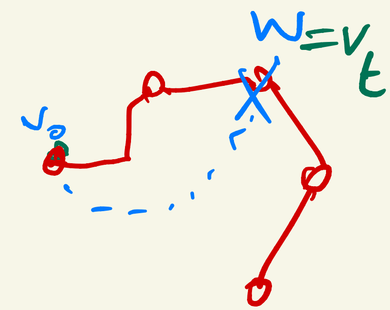


Proof. Let $v_0 - \dots - v_\ell$ be a longest path in T (tree)

Claim $\deg(v_0) = 1$

P — longest path





Pf by contradiction.

$\deg(v_0) \neq 0$ b/c $n \geq 2$

$\therefore$ $\exists$ path from v_0 to some other vertex

NTS $\deg(v_0) \geq 2$ impossible

Assume for a contradiction that $\exists w \neq v_1$, $w \sim v_0$

Case 1 $\quad w \notin V(P) \implies w, v_0, v_1, \ldots, v_\ell$ is a path of length $\ell + 1$ $\rightarrow\leftarrow$

Case 2 $\quad w \in V(P) \implies v_0 \, w = v_t \, v_{t-1} \cdots v_2 \, v_1$ is a cycle $\rightarrow\leftarrow$

Thm If T is a tree then $m = n-1$

size — m, order — n

Proof. Induction on n.

Base case: $n=1$ $\quad\circ\quad$ $m=0$ ✓

Now assume $n \geq 2$

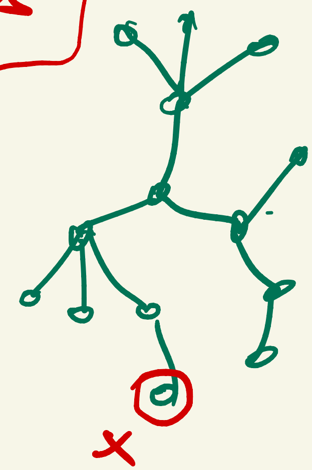
Suppose Thm true for $\forall n' < n$ ← $\boxed{\text{Inductive Hypothesis}}$

NTS: true for n

need to construct a tree T' with $n' < n$ vertices to which we intend to apply IH

T: tree of order n

Let x be a pendant vertex of T



Let $T' := T - x$ (remove x and the edge <u>incident</u> with x)

T' has order $n' = n-1$
size $m' = m-1$

$\underline{\text{by IH}}$ applied to T'

$$m' = n' - 1$$

$$\begin{array}{l} m-1 = n-1-1 \qquad +1 \\ \hline m = n-1 \end{array}$$

✓

I.H. If T' is a tree need to verify
with $n' < n$ vertices
then $m' = n' - 1$

$\boxed{\text{HW}}$ Lemma If S is a tree and y is a pendant vertex of S
then $S - y$ is a tree