PROBLEM SESSION | 2023-10-20 | 1

PREDICATE on set A is $f: A \to \{0, 1\}$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ F $\;$ T

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ N $\;$ Y

predicates on A : 2^n

$|A| = n$

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relation from A to B : predicate on $A \times B$

$|A| = m \quad |B| = k$

$R(a,b) = \begin{cases} 0 \\ 1 \end{cases}$

#relations from A to B : 2^{nk}

$|A \times B| = nk$

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homogeneous relation: $B = A$

$A \times A$

$A = \mathbb{Z}$

rel: $a \leq b$

$a \mid b$ divisor

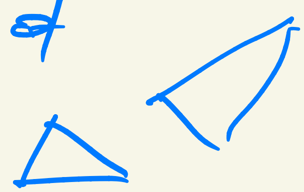
$a \equiv b \mod m$

"$R(a,b) = 1$"

shorthand

aRb

(Similarity of triangles



DEF R is reflexive if $(\forall x \in A)(xRx)$

not —"— if $(\exists x \in A)(x \not R x)$

$(\neg xRx)$

DEF R is irreflexive if $(\forall x \in A)(x \not R x)$

Can R be both reflexive and irreflexive ?

such R exists $\iff A = \emptyset$

$|A| = n$ # reflexive relations

$n^2 - n$ binary decisions left

$\therefore$ #: 2^{n^2-n}

irreflexive relations

Same: 2^{n^2-n}

relations on A: 2^{n^2}

$$\begin{array}{c|cccc} & a & b & c & d \\ \hline a & 1 & & & \\ b & & 1 & 1 & 0 \\ c & & & 1 & \\ d & & & & 1 \end{array}$$

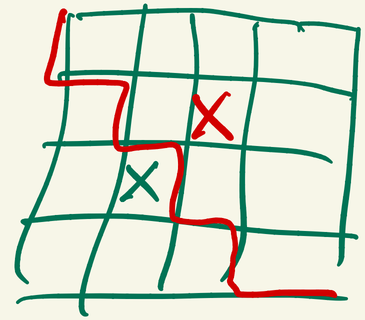
bRc

$b \not R d$

$$\begin{array}{cccc} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{array}$$

R is symmetric: $(\forall x,y \in A)(xRy \Rightarrow yRx)$

#symmetric relations $\quad 2^{\frac{n(n+1)}{2}}$



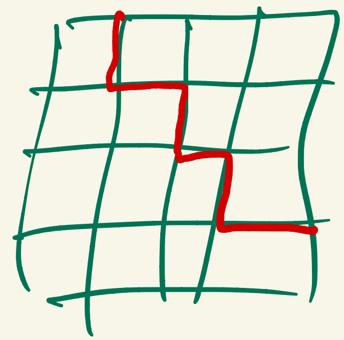
#decisions

$S = \underline{n + (n-1) + \cdots + 1} = \frac{n(n+1)}{2}$

$\underline{S = 1 + 2 + \cdots + n}$

$2S = (n+1) + (n+1) + \cdots + (n+1) = n(n+1)$

#symmetric, reflexive relations $\quad 2^{\binom{n}{2}}$



#decisions

$(n-1) + (n-2) + \cdots + 1 = \frac{n(n-1)}{2}$

#2-subsets $\quad \binom{n}{2} = \frac{n(n-1)}{2}$

another reasoning:

We need to make a decision for each <u>unordered</u> pair $\{a,b\}$ of distinct elements $a \neq b$

2-subsets

\# graphs on a given set of n vertices

$|V| = n$

adjacency is $\begin{cases} \text{irreflexive} \\ \text{symmetric} \end{cases}$

$\therefore \binom{n}{2}$ decisions $\rightsquigarrow 2^{\binom{n}{2}}$ graphs

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DEF $B(n)$: # partitions of an n-set

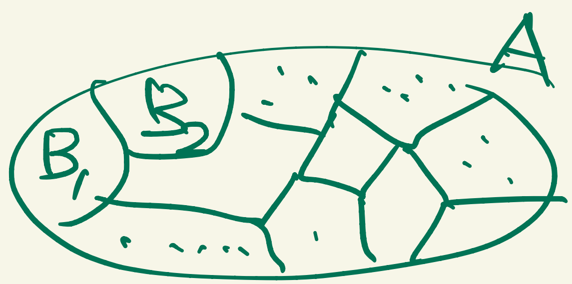
DEF Partition of A $\quad \Pi = \{B_1, \dots, B_k\}$ — blocks of the partition

s.t. (0) $B_i \subseteq A$

(i) $B_i \neq \emptyset$

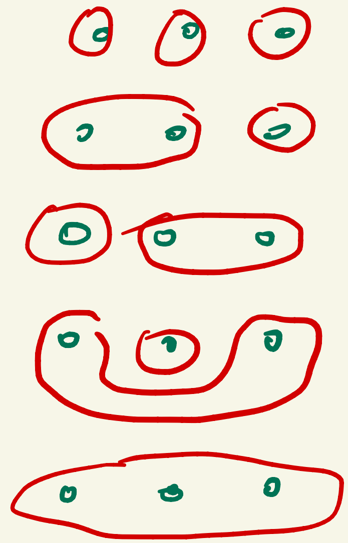
(ii) $i \neq j \Rightarrow B_i \cap B_j = \emptyset$ disjoint

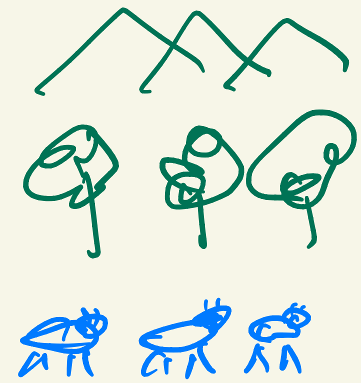
(iii) $\bigcup_{i=1}^{k} B_i = A$



THM # equiv. relations on A is $B(n)$

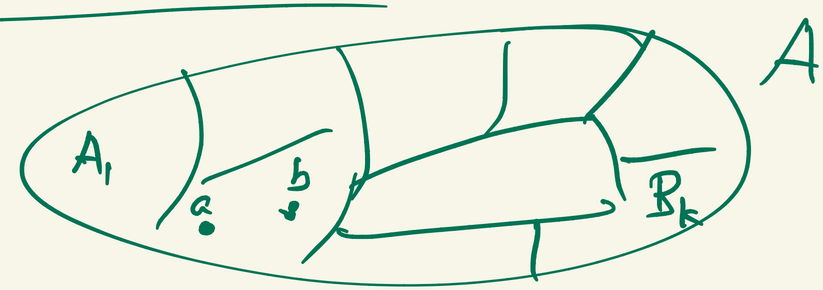
$B(3) = 5$





$B(n) \le n!$

Π



domain: partitions

codomain: equivalence relations

$a \, R_\Pi \, b$ iff $(\exists i)(a, b \in B_i)$

$\boxed{\Pi \longmapsto R_\Pi}$ equiv. rel.

DO injective: If $R_\Pi = R_{\Pi'}$ then $\Pi = \Pi'$

! Thm surjective: $(\forall \text{ eq. rel } S \text{ on } A)(\exists \text{ partition } \Pi)(S = R_\Pi)$

Pf of FUND. THM of EQ.REL's

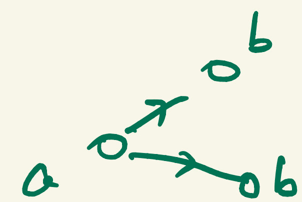
NTS $\exists$

Input: Eq. rel R Desired output: partition Π s.t. $R = R_\Pi$

$\Pi = \{B_1, \dots, B_k\}$

for $a \in A$

- $[a] := \{b \in A \mid aRb\}$



Claim: the set $[a]$ partition A

(a) $[a] \subseteq A$

(b) $[a] \neq \emptyset \Leftarrow$ (b1) $a \in [a]$ reflexive: aRa ✓

(d) $\bigcup_{a \in A} [a] = A$ NTS $\bigcup [a] \subseteq A \Leftarrow (\forall a)([a] \subseteq A)$

$\supseteq A$: NTS $c \in A \Longrightarrow c \in \bigcup [a]$

$x \in D_1 \cup \dots \cup D_k \Longleftrightarrow (\exists i)(x \in D_i)$

NTS $(\exists a)(c \in [a])$

$\uparrow$

$a := c$ $c \in [c]$

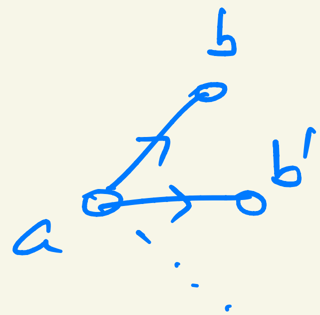
(c) $(\forall a,b)([a]=[b] \vee [a] \cap [b] = \emptyset)$

$$\underbrace{(\forall a,b \in A)\big([a]=[b] \vee [a]\cap[b]=\emptyset\big)}$$

$$[a]\cap[b]\neq\emptyset \implies [a]=[b]$$

ASSN $[a]\cap[b]\neq\emptyset$

DC $[a]=[b]$



$[a]=\{b\in A \mid \underline{aRb}\}$

$\exists\ c\in[a],\ c\in[b]$

Lemma 1 $x\in[y] \implies y\in[x]$

i.e. $yRx \implies xRy$ b/c R is symmetric

$\therefore\ a,b\in[c]$

Lemma 2 $x\in[y] \implies [x]\subseteq[y]$

ASSN $\underline{yRx}$ D.C. $(\forall z)(z\in[x] \implies z\in[y])$

$\underline{xRz} \overset{?}{\implies} \underline{yRz}$ transitivity

NTS $[a] = [b]$

NTS $\underline{[a] \subseteq [b]}$ and $[b] \subseteq [a]$

initial assn:

$c \in [a]$, $c \in [b]$

$\Downarrow$ Lemma 1 $\quad\quad$ $\Downarrow$ Lemma 2

$a \in [c]$ $\quad\quad$ $[c] \subseteq [b]$

$\therefore\ a \in [b]$

Lemma 2 $x \in [y] \Rightarrow [x] \subseteq [y]$ ✓

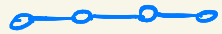
NTS $a \in [b]$

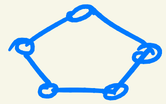
b/c Lemma 2

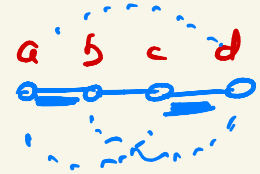
$\therefore\ [a] \cap [b] \neq \emptyset \Rightarrow [a] \subseteq [b]$

$[a] \cap [b] \neq \emptyset \Rightarrow [b] \subseteq [a]$ by swapping the roles of $\underline{a}$ and $\underline{b}$

$n=1$  K_1

$n=4$  P_4

$n=5$  C_5



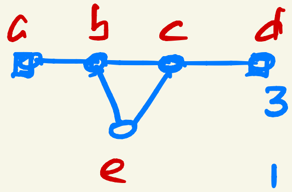
$a \mapsto c$

$b \mapsto a$

$c \mapsto d$

$d \mapsto b$

\mapsto



$$\left.\begin{array}{lllll} 3 & 1 & a & \mapsto & c \\ 1 & 3 & b & \mapsto & a \\ 1 & 3 & c & \mapsto & d \\ 3 & 1 & d & \mapsto & b \end{array}\right\}$$

$$\begin{array}{lllll} 2 & 2 & e & \mapsto & \underline{\underline{e}} \end{array}$$

If $\exists$ self-compl. graph of order n then $\underline{n \equiv 0 \text{ or } 1 \bmod 4}$

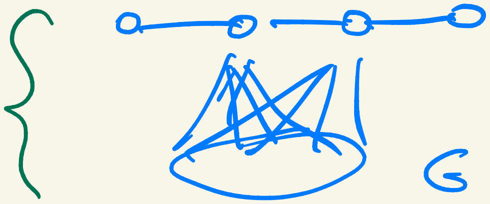
$G = (V, E)$ $\qquad |E| + |\bar{E}| = \binom{n}{2} = \frac{n(n-1)}{2}$

$\bar{G} = (V, \bar{E})$

$G \cong \bar{G} \Rightarrow |E| = |\bar{E}| = \frac{n(n-1)}{4}$

$\therefore\ 4 \mid n(n-1)$ $\qquad \begin{cases} \text{if } n = \text{odd} \Rightarrow 4 \mid n-1 & \text{i.e. } n \equiv 1 \ (4) \\ \text{if } n-1 = \text{odd} \Rightarrow 4 \mid n & n \equiv 0 \ (4) \end{cases}$

thm $(\forall n)(n \equiv 0 \text{ or } 1 \mod 4 \Longrightarrow \exists$ self-compl. graph of order $n)$

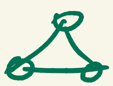
H 

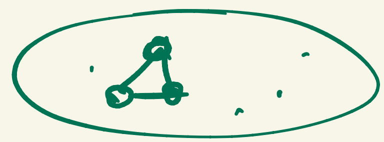
G n

$G \cong \overline{G}$

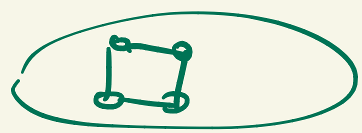
$\boxed{\text{DO}}$ $H \cong \overline{H}$

\# triangles in K_n $\binom{n}{3}$

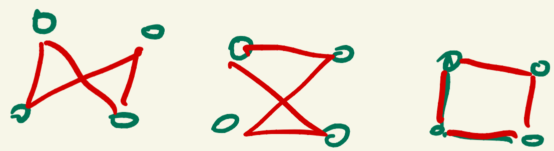




\# 4-cycles C_4



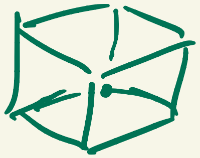
$3 \cdot \binom{n}{4}$



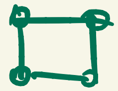
Q_d d-dim cube

\# vertices

$$|V(Q_d)| = 2^d$$

Q_3 

$$V(Q_d) = \{0,1\}^d$$

$Q_2 = C_4$ 

$Q_1 = K_2$ 

$Q_0 = K_1$ 

$\boxed{\text{degree} = d}$

00000
00001
00010
00011
00100
00101
00110
00111
⋮
11011 even # 1s
10011 odd # 1s

<u>bipartite</u> NTS $\exists$ partition $V(Q_d) = A \,\dot{\cup}\, B$ (disjoint)

<u>s.t.</u> $\forall$ edge goes between A and B

HW G d-regular

\# walks of length k

\# sequences $(v_0, \dots, v_k)$ s.t. $(\forall i)(v_{i-1} \sim v_i)$

~~8.31~~