2023-10-24 | 1

$\binom{n}{k}$ = # k-subsets of an n-set

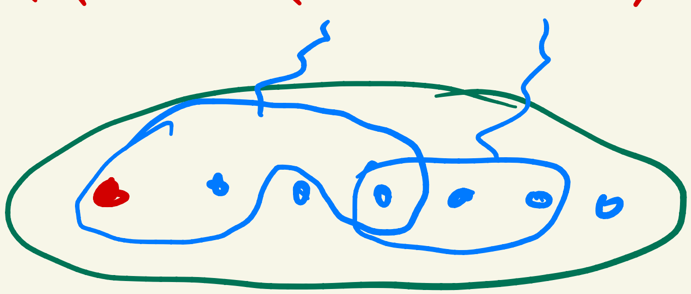
$n, k \geq 0$ integers

if $k > n$ then $\binom{n}{k} = 0$

$\binom{n}{0} = 1$ $\binom{n}{n} = 1$

PASCAL'S IDENTITY

$$\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$$



$n+1$

PASCAL's triangle

#0 $\binom{0}{0}$

#1 $\binom{1}{0}$ $\binom{1}{1}$

#2 $\binom{2}{0}$ $\binom{2}{1}$ $\binom{2}{2}$

$$\ldots \binom{n}{k-1} \quad \binom{n}{k} \quad \binom{n}{k+1} \ldots$$

$$\binom{n+1}{k} \quad \binom{n+1}{k+1}$$

BINOMIAL THEOREM

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

$(x+y)^0 = 1$

$(x+y)^1 = x+y$

$(x+y)^2 = x^2 + 2xy + y^2$

$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

Proof

$(x+y)^n = (x+y)(x+y)\cdots(x+y)$

2^n expansion terms

each term has the form

$\binom{n}{k} x^k y^{n-k}$

#terms

DIRECT PROOF USING DEF of $\binom{n}{k}$

$y\,x \dots y$

$n=3$

$(x+y)(x+y)(x+y)$

repeat

2nd proof: induction on $\underline{\underline{n}}$

Base case: true for $n=0$ $\quad 1 = (x+y)^0 = \sum_{k=0}^{0} \binom{0}{0} x^k y^{n-k} = 1 \cdot x^0 y^0 = 1$ ✓

Now assume $n \geq 1$

I.H. Thm true for $\forall n' \leq n-1$

DC Thm for $\underline{\underline{n}}$

$$(x+y)^n = \underline{\underline{(x+y)^{n-1}}} \cdot (x+y) = \left[\overbrace{\sum_{k=0}^{n-1} \binom{n-1}{k} x^k y^{n-k-1}}^{IH}\right](x+y) =$$

$$= \sum_{k=0}^{n-1} \binom{n-1}{k} x^{k+1} y^{n-k-1} + \sum_{k=0}^{n-1} \binom{n-1}{k} x^k y^{n-k}$$

cases $k=0$ and $k=n$ checked seperately

$$= \sum x^k y^{n-k} \cdot \left(\overbrace{\binom{n-1}{k} + \binom{n-1}{k-1}}^{\text{PASCAL'S ID.}}\right) = \sum_{k=0}^{n} x^k y^{n-k} \cdot \binom{n}{k}$$

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

$x := 1$

$$2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k}$$

plug $x := 1$ into Bin. Thm

$$\boxed{\sum_{k=0}^{n} \binom{n}{k} = 2^n}$$ ← "algebra proof"

2nd pf: Combinatorial proof:

LHS: # all subsets of n-set

RHS: $= |\mathcal{P}([n])| = 2^n$

pf: bijection $\mathcal{P}([n]) \to \{0,1\}^n$

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$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

$x := -1$

$$0^n = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} \pm \cdots$$

$$0^n = \begin{cases} 0 & \text{for } n \geq 1 \\ 1 & \text{for } n = 0 \end{cases}$$

$\leftarrow$ empty product $= 1$

like $0! = 1$

		1			1	$= 1$
	1		1		$1 - 1$	$= 0$
1		2		1	$1 - 2 + 1$	$= 0$
1	3		3	1	$1 - 3 + 3 - 1$	$= 0$
1	4	6	4	1	$1 - 4 + 6 - 4 + 1$	$= 0$

thm $\binom{n}{0}-\binom{n}{1}+\binom{n}{2}-+\cdots=\begin{cases}1 & \text{if } n=0\\ 0 & \text{if } n\geq 1\end{cases}$ 6

algebra pf: plug in $x:=-1$ into BIN THM

Combinat pf.

$$E_n := \sum_{k=0}^{\lfloor n/2 \rfloor}\binom{n}{2k} = \#\text{ even subsets}$$

$$O_n := \sum_{k=0}^{\lfloor (n-1)/2 \rfloor}\binom{n}{2k+1} = \#\text{ odd ---''---}$$

thm restated:

$$E_n - O_n = \begin{cases}1 & \text{if } n=0\\ 0 & \text{if } n\geq 1\end{cases}$$

$\boxed{\text{HW}}$ Find a simple combinatorial pf: bijective pf

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$$S(n,t) := \sum_{k=0}^{\infty} \binom{n}{tk}$$ sum up to $\lfloor \frac{n}{t} \rfloor$

$$= \sum_{k=0}^{\lfloor \frac{n}{t} \rfloor} \quad \text{if } t \geq 1$$

$$S(n,0) = \infty$$

$$S(n,1) = 2^n = \frac{2^n}{1}$$

$$S(n,2) = E_n = \begin{cases} 1 & n=0 \\ 2^{n-1} = \frac{2^n}{2} & n \geq 1 \end{cases}$$

$$E_n + O_n = 2^n$$
$$E_n = O_n$$
$$E_n = \frac{2^n}{2} = 2^{n-1}$$

$$S(n,3) \overset{?}{=} \frac{2^n}{3}$$ NO: never an integer

pattern?

BUT

$$S(n,3) = \binom{n}{0} + \binom{n}{3} + \binom{n}{6} + \dots$$

XC

$$\left| S(n,3) - \frac{2^n}{3} \right| < 1$$