$[n] = \{1, 2, \dots, n\}$

$[3] = \{1, 2, 3\}$

$[1] = \{1\}$

$[0] = \emptyset$

$f : [k] \longrightarrow [m]$

monotone increasing: $f(1) < f(2) < \dots < f(k)$

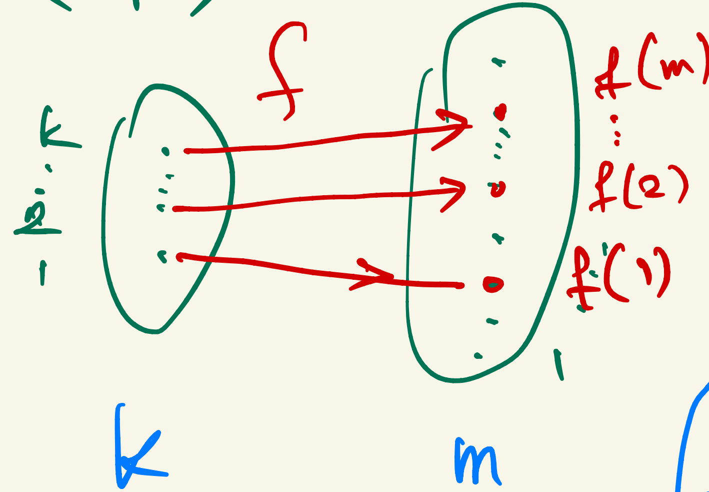
-"- nondecreasing: $f(1) \le f(2) \le \dots \le f(k)$

increasing fctns: $\mathrm{incr}(k, m) = ?$

2

$f : [k] \to [m]$

\# increasing fctns: $\mathrm{incr}(k,m)$



bijection

$\{\text{increasing fctns}\} \to \binom{[m]}{k}$

bijection

$\therefore\ \boxed{\mathrm{incr}(k,m) = \binom{m}{k}}$

Example

$k=3 \quad m=5$

1	1	1	1	1	1	1	2	2	2	3
2	2	2	2	3	3	4	3	3	4	4
3	3	4	5	4	5	5	4	5	5	5

1	$f(1)$
2	$f(2)$
$\vdots$	$\vdots$
k	$f(k)$

$1 \le f(i) \le m$

pick 3-subset of 5

$\{4, 1, 3\}$

10 $\quad \mathrm{incr}(3,5) = 10$

$\binom{5}{3} = \binom{5}{2} = \frac{5\cdot 4}{2} = 10$

$\mathrm{nondecr}(k, m) = ?$

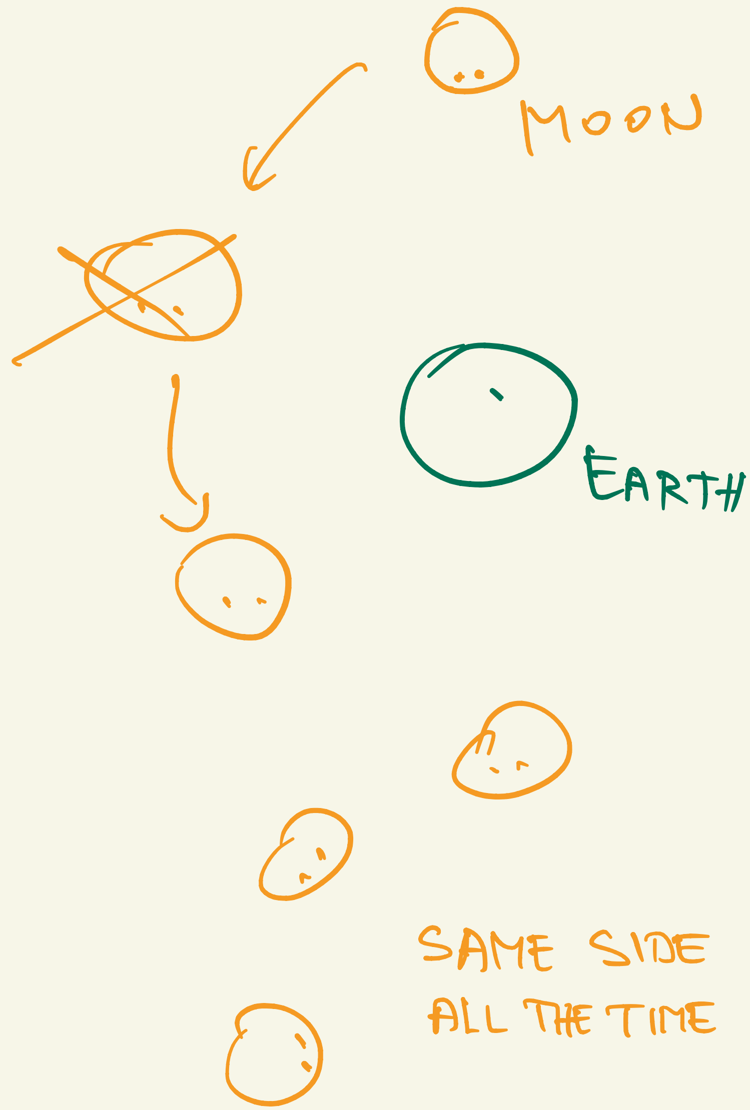
Example

$k = 3, \quad m = 3$

1	1	1	1	1	1	1	2	2	2	3
2	1	1	1	2	2	3	2	2	3	3
3	1	2	3	2	3	3	2	3	3	3

10

$\mathrm{nondecr}(3, 3) = \mathrm{incr}(3, 5)$



thm $\text{nondecr}(k,m) = \text{incr}(k, m+k-1)$

$$1 \le f(1) \le \dots \le f(k) \le m \longmapsto 1 \le g(1) < g(2) < \dots < g(k) \le m+k-1$$

bijective proof

$$\boxed{g(i) := f(i) + (i-1)}$$

$\boxed{\text{DO}}$ f is nondecr. $[k] \to [m] \iff g$ is incr. $[k] \to [m+k-1]$

f: nondecr. $[3] \to [3] \longmapsto g$: incr $[3] \to [5]$

EXAMPLE

	f	g
1	1+0	1
2	1+1	2
3	1+2	3

3,3

f	g
1+0	1
1+1	2
3+2	5

3,5

Given g find f

1 − 0	1
2 − 1	1
5 − 2	3

f	g
1 +0	1
1 +1	2
1 +1	2

5

$$\mathrm{incr}(k,m) = \binom{m}{k}$$

$$\therefore \quad \mathrm{nondecr}(k,m) = \binom{m+k-1}{k}$$

$k := k$

$m := m+k-1$

(a) $\boxed{\text{HW}}$ $\left\{ \begin{array}{l} \sum_{i=1}^{\ell} x_i = n, \quad x_i \in \mathbb{N} = \{1,2,3,\dots\} \\ \text{Q}: \#\text{solutions}? \end{array} \right.$

proof: bijection

{set of sol's} $\longrightarrow$ {incr. fctns}

find k, m in terms of ℓ, n

(b) $\boxed{\text{HW}}$ $\sum_{i=1}^{\ell} y_i \le n$

same type of bijection

$\hookrightarrow$ {nondecr. fctn}

(c) $\boxed{\text{HW}}$ same as (a) but $x_i \ge 0$

(d) $\boxed{\text{HW}}$ — " — (b) — " — $y_i \ge 0$ — " —

$\boxed{\text{HW}}$ f strongly increasing: $f(i+1) \geq f(i) + 2$

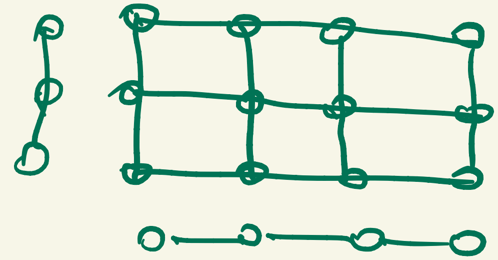
Count ——— - fcts $[k] \to [m]$

---

bijection

answ: simple binomial coefficient

Graph theory

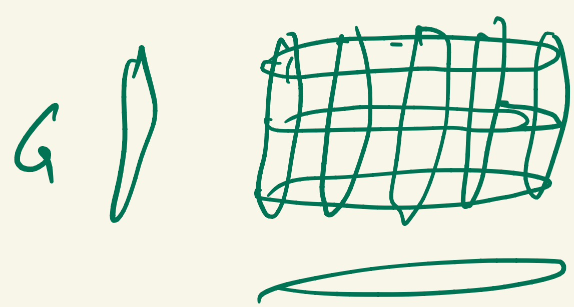


$G = (V, E)$

$H = (W, F)$

$J = G \square H$ Cartesian product of G and H

$V(J) := V \times W$



$= \{(v, w) \mid v \in V, w \in W\}$

adjacency

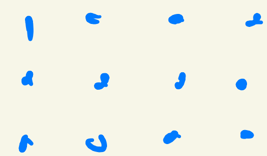
$(v_1, w_1) \sim_J (v_2, w_2)$ if

$v_1 = v_2$ and $w_1 \sim_H w_2$

OR

$w_1 = w_2$ and $v_1 \sim_G v_2$

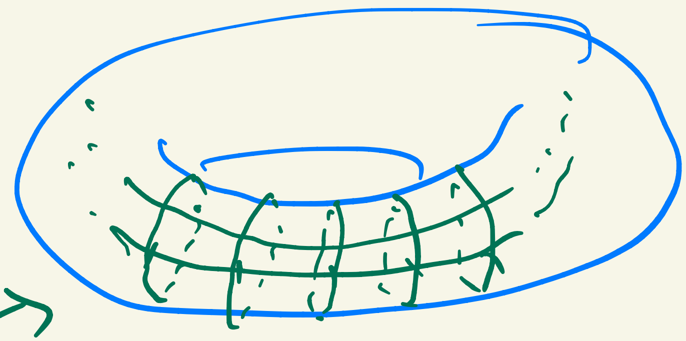
$V(J)$



$P_k \square P_\ell$: $k \times \ell$ grid

$C_k \square C_\ell$

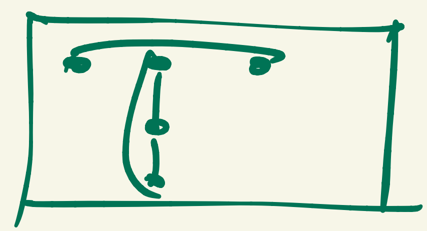
torus

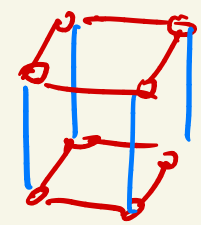


$k \times \ell$ toroidal grid

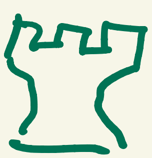
$K_k \square K_\ell$

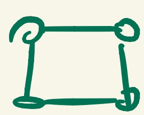
rook graph





Q_3

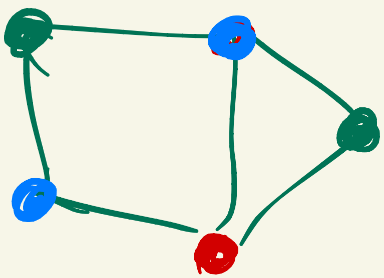




$K_2 \square K_2 = C_4 = Q_2$

$K_2 \square K_2 \square K_2$

$$\underbrace{K_2 \square \cdots \square K_2}_{d \text{ copies}} = Q_d \quad d\text{-dim. cube}$$



Legal coloring of $G = (V, E)$:

$f : V \to \{\text{colors}\}$

s.t. if $v \sim w \implies f(v) \neq f(w)$

min # colors needed for a legal coloring:

chromatic number $\chi(G)$

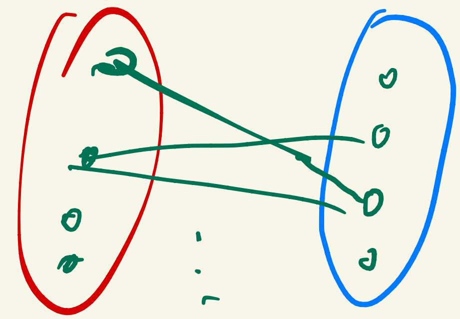
\chi

G is k-colorable if k colors suffice, i.e. $\chi(G) \leq k$

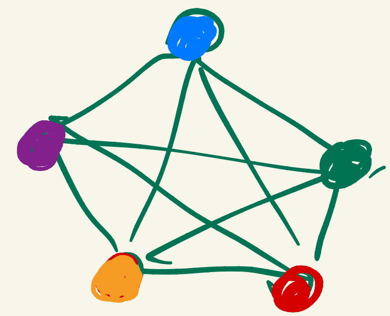
2-colorable graphs

$\Updownarrow$

bipartite



$\chi(K_n) = n$



$(\forall G)(\chi(G) \leq n)$

$\boxed{\text{HW}}$ $\chi(C_n)$ $n \geq 3$

$\chi(\text{tree})$