PROBLEM SESSION

2023.10.27

Geometric mean → arithmetic mean

$\sqrt{ab} \leq \frac{a+b}{2}$

8.43 # sp. subgraphs of K_n with m edges

$\binom{\binom{n}{2}}{m}$

$|E(K_{r,n-r})| = r(n-r) \leq \frac{n^2}{4} \Leftarrow ab \overset{?}{\leq} \left(\frac{a+b}{2}\right)^2$

i.e. $4ab \overset{?}{\leq} (a+b)^2 = a^2+b^2+2ab$

i.e. $0 \overset{?}{\leq} a^2+b^2-2ab = (a-b)^2 \geq 0$

$\therefore |E(K_{r,n-r})| \leq \lfloor \frac{n^2}{4} \rfloor$

$(\exists r)\, |E(K_{r,n-r})| = \lfloor \frac{n^2}{4} \rfloor$

$r := \lfloor \frac{n}{2} \rfloor$

Case 1: n even: $r = \frac{n}{2}$ $\quad r(n-r) = \frac{n^2}{4}$ ✓

Case 2: n odd: $r = \frac{n-1}{2}$ $\quad r(n-r) = \frac{n-1}{2} \cdot \frac{n+1}{2} = \frac{n^2-1}{4} = \frac{n^2}{4} - \frac{1}{4} = \lfloor \frac{n^2}{4} \rfloor$ ✓

8.38 max # edges of bipartite graph $= \lfloor \frac{n^2}{4} \rfloor$

$\lfloor x \rfloor$ rounded-down value: $\lfloor 5.2 \rfloor = 5$

$\lfloor \pi \rfloor = 3$

$\lfloor -5.2 \rfloor = -6$

$\lfloor 4 \rfloor = 4$

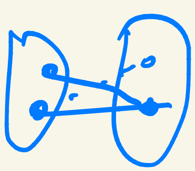
floor fctn

If $a \in \mathbb{Z}$, $x \in \mathbb{R}$

$a \leq x$

$\Downarrow$

$a \leq \lfloor x \rfloor$



$K_{r,n-r}$ $\quad |E(K_{r,n-r})| = r(n-r)$

$\max_{1 \leq r \leq n-1} r(n-r) = \frac{n}{2} \cdot \left(n - \frac{n}{2}\right) = \frac{n^2}{4}$

Suppose we want to show: $\chi(G) = 5$

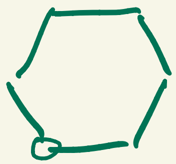
difficult ① $\chi(G) \geq 5$: $\forall$ legal coloring uses ≥ 5 colors

easy ② $\chi(G) \leq 5$: $\exists$ legal coloring with colors

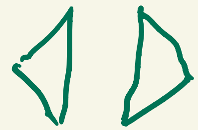
(show a coloring)

8.49 (a)

G



H



reg deg $= 2$

$n = 6$

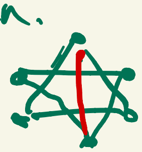
not isomorphic b/c $C_3 \subseteq H$ $C_3 \not\subseteq G$

H is disconn. G conn.

(b) make them connected

$\bar{G}$ $\bar{H}$ ✓

NTS: conn.



✓

deg 3

$n = 6$

both conn.

$\bar{G} \neq \bar{H}$ b/c $G \not\cong H$

9.27 T tree, x pendant vx: $\deg x = 1$
$\Longrightarrow T - x$ tree

NTS $T-x$ is a tree, i.e. $T-x$ has no cycles (a)
$T-x$ is conn. (b)

(a) obvious: suppose for a contradiction that

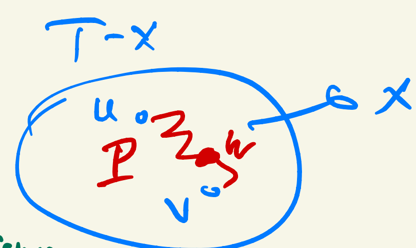
$$\underbrace{C \subseteq T - x \subseteq T}_{\text{transitivity of "subgraph" relation}} \Longrightarrow C \subseteq T \rightarrow\leftarrow$$

cycle

T is cycle free

(b) Let $u, v \in V(T-x)$
$= V(T) \setminus \{x\}$

i.e. $u, v \in V(T)$, $u, v \neq x$

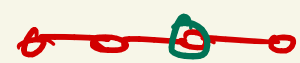


DEF of connectedness $\Rightarrow \boxed{\text{NTS}}$ $\exists\, u \cdots v$ path in $T-x$

by ASSUMPTION: T is conn.

We know: $\exists\, u \cdots v$ path in T

Q: Prove $x \notin V(P)$



take an interior vertex w of $P \Longrightarrow \deg_P(w) = 2$

$\therefore \deg_T(w) \geq 2$ $\therefore w \neq x$ ASSN x is pendant

now, take an endpoint of P: z Can $z = x$? No $u, v \neq x$ b/c $u, v \in V(T-x)$

8.13 $F_0=0$ $F_1=1$ $F_2=1$ $F_3=2$ $F_4=3$ $F_5=5$ $F_6=8$

Claim for $n\ge 6$ $F_n \ge 2^{n/2}$

Pf by induction on n

Base case(s): $n=6,7$

$n=6$ $F_6 = 8 = 2^3 = 2^{6/2}$ ✓

$n=7$ $F_7 = 13 \overset{?}{\ge} 2^{7/2}$

$F_7^2 = 169 \ge 128 = 2^7$ ✓

Inductive step

Inductive Hypothesis: Thm true for all $n' < n$

IC: Thm for n

~~Thm $\forall n$~~

NTS for the current value of n $F_n \ge 2^{n/2}$

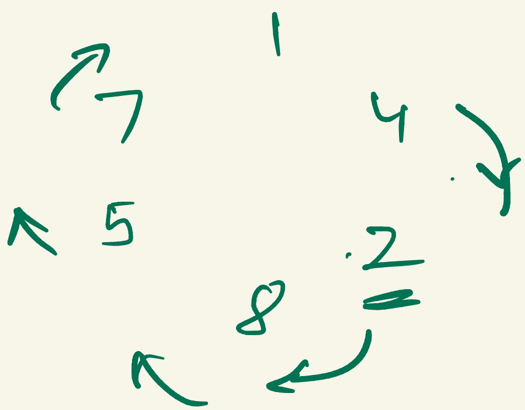
$$F_n = F_{n-1} + F_{n-2} \ge 2^{\frac{n-1}{2}} + 2^{\frac{n-2}{2}} \ge 2\cdot 2^{\frac{n-2}{2}} = 2^{n/2}$$ ✓

by I.H.

Need: $n-2 \ge 6$ i.e. $n \ge 8$

$\frac{1}{7} = 0.\dot{1}4285\dot{7}$

$2^{20} = 1,048,576$



$\frac{1}{7} = 0.\dot{1}4285\dot{7}$

$A = 142857$

$2A = 285714$

$3A = 428571$

$4A = 571428$

$5A = 714285$

$6A = 857142$

$7A = 999999$

8.31 $\quad n+1 \mid \binom{2n}{n}$

$$\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$$

b/c $\dfrac{\boxed{n}(n-1)\cdots(n-k+1)}{\boxed{k}(k-1)\cdots 1}$

EUCLID'S LEMMA: If p prime and $p \mid ab$ then $p \mid a \vee p \mid b$

GENERALIZATION: If $k \mid ab$ and $\gcd(k,a)=1$ then $k \mid b$

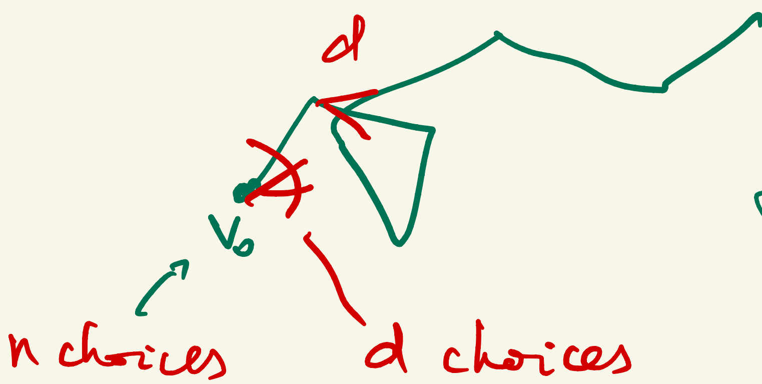
$$\binom{2n}{n} = \frac{(2n)(2n-1)\cdots\boxed{(n+1)}}{1\cdot 2\cdot\cdots\boxed{n}} = \frac{n+1}{n}\cdot\binom{2n}{n-1}$$

$$\therefore\ n\binom{2n}{n} = (n+1)\cdot\binom{2n}{n-1}$$

$\therefore\ \underline{n+1} \mid \underline{n}\cdot\binom{2n}{n}$ but $\gcd(n+1,n)=1$ $\therefore\ n+1 \mid \binom{2n}{n}$ ✓

relatively prime

9.51 # walks of length k in d-regular graph



$n \cdot d^k$

3.42 $\binom{n}{k} < \left(\frac{ne}{k}\right)^k$ $\qquad \binom{n}{k} \ge \left(\frac{n}{k}\right)^k$ ✓

Pf Lemma 1 $n! > \left(\frac{n}{e}\right)^n$ Pf $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ ← plug in $x := k := n$

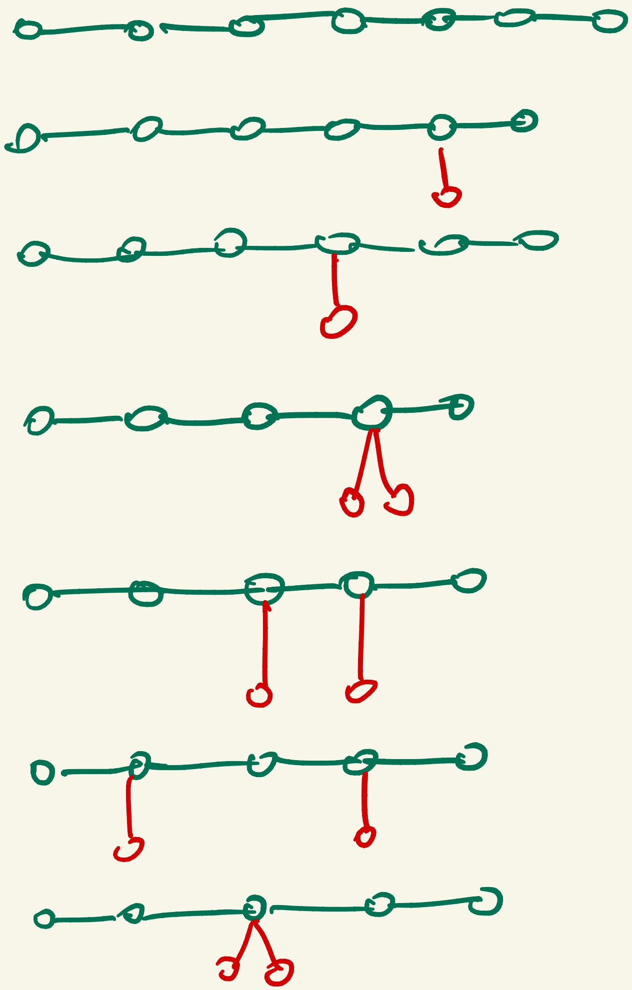
$e^n > \frac{n^n}{n!}$

Lemma 2 $\binom{n}{k} \le \frac{n^k}{k!}$ Pf $\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!} \le \frac{n^k}{k!}$

$\therefore \binom{n}{k} \underset{L2}{\le} \frac{n^k}{k!} \underset{L1}{\le} \frac{n^k}{\left(\frac{k}{e}\right)^k} = \left(\frac{en}{k}\right)^k$ ✓

8.51 trees with 7 vertices

organize by longest path



1 — P_7 — 6

2 — 5

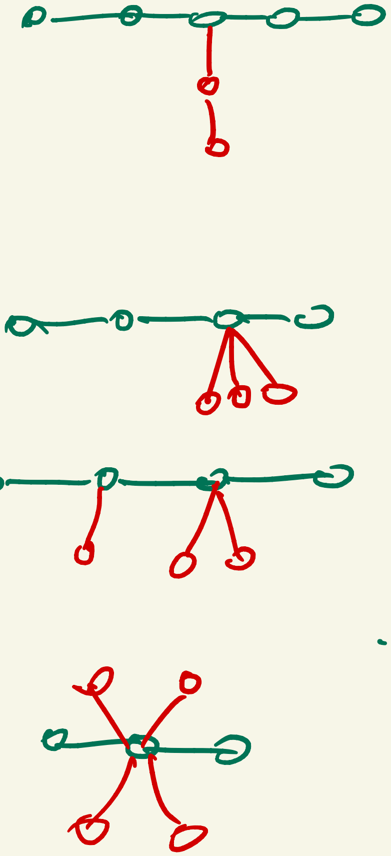
3

4

5

6

7



5 — 8

4 — 9

3 — 10

2 — 11

8.15 $(\forall n \geq 2)\left(F_{n+2} \overset{?}{=} 3F_n - F_{n-2}\right)$

not by induction

$$\underline{F_{n+1} + F_n} \overset{?}{=} 3F_n - (F_n - F_{n-1}) \overset{!}{=} \underline{2F_n + F_{n-1}}$$

$-F_n$

$$\Uparrow$$

$$F_{n+1} = F_n + F_{n-1}$$

8.17 $\quad F_n^2 - F_{n+1} \cdot F_{n-1} \overset{?}{=} (-1)^{n+1}$

inductive step

← prove for current value of $\underline{\underline{n}}$

← based on smaller values of $\underline{\underline{n}}$

I.H. $\quad F_{n-1}^2 - F_n F_{n-2} \overset{!}{=} (-1)^n$

add them: $\quad F_n^2 + F_{n-1}^2 - F_{n+1} F_{n-1} - F_n F_{n-2} \overset{?}{=} 0$

$$F_n^2 - F_n F_{n-2} - F_{n+1} F_{n-1} + F_{n-1}^2 \overset{?}{=} 0$$

$$F_n \underbrace{(F_n - F_{n-2})} - F_{n-1} \cdot \underbrace{(F_{n+1} - F_{n-1})} \overset{?}{=} 0$$

$$F_n \cdot F_{n-1} - F_{n-1} \cdot F_n \overset{!}{=} 0$$

LOGIC

Base case: $\quad n = 1 \qquad F_1^2 - F_2 F_0 \overset{?}{=} (-1)^2$

$$1 - 1 \cdot 0 = 1 \quad ✓$$

Case missed by inductive step b/c it involves F_{n-2}

- $$F_{n+1} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} = 1 + \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k}$$

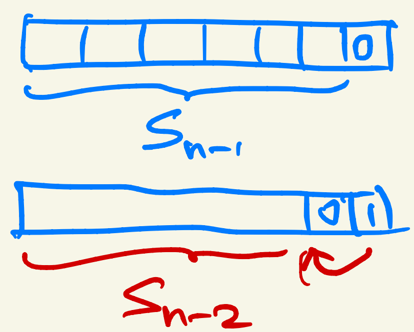
take separately case of $k=0$

Pf by induction using Pascal's identity

$F_n + F_{n-1}$

3.24 # (0,1)-strings of length n without consecutive 1s : S_n

Claim $S_n = S_{n-1} + S_{n-2}$ $\quad n \geq 2$



base case
$S_0 = 1 = F_2$
$S_1 = 2 = F_3$

$S_2 = 3 = F_4$

Claim $S_n = F_{n+2}$

Pf by induction

F_0	F_1	F_2	F_3	F_4
0	1	1	2	3