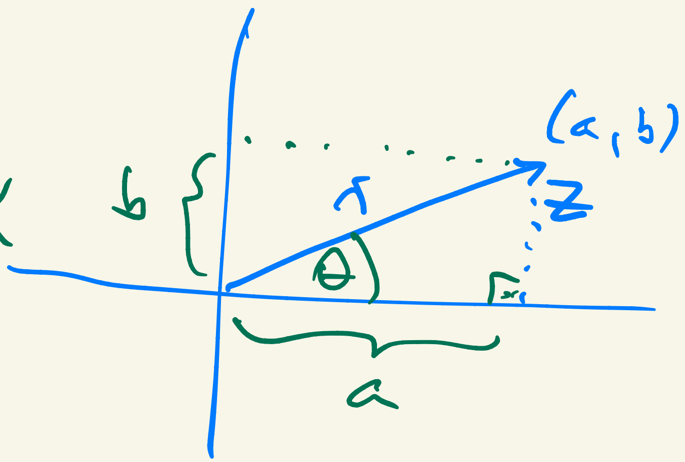
COMPLEX NUMBERS

10-31-2023 | 1

canonical form

$z = a + bi$

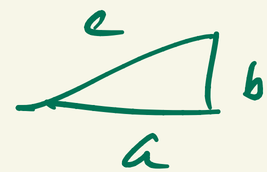
$a, b \in \mathbb{R}$ $\qquad i^2 = -1$



(a, b) represents $a+bi$

$r = |z|$ modulus, absolute value

$r = \sqrt{a^2+b^2}$ Pythagorean Theorem

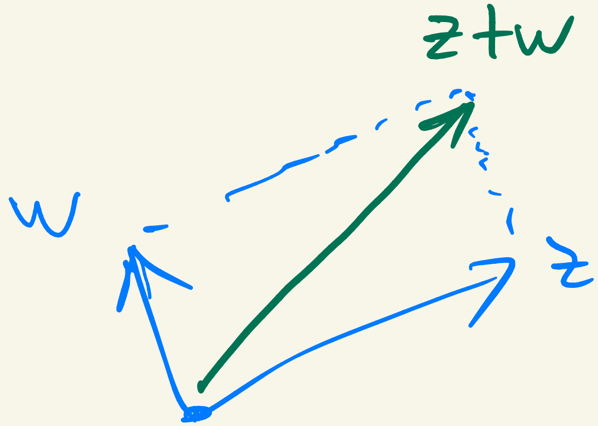


$c^2 = a^2 + b^2$

$\frac{z}{|z|} = \cos\theta + i\sin\theta$

$$\boxed{z = r(\cos\theta + i\sin\theta)}$$

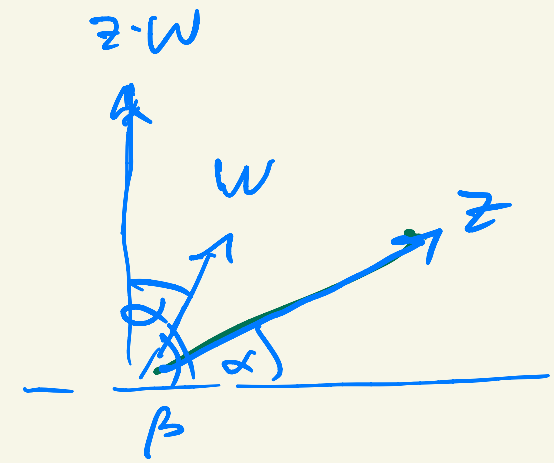
polar form



$$z = a + bi$$
$$w = c + di$$
$$\overline{\quad\quad\quad\quad}$$
$$z + w = (a+c) + (b+d)i$$

$$z \cdot w = (a+bi)\cdot(c+di) = (ac - bd) + (ad + bc)i$$

$$z = r(\cos\alpha + i\sin\alpha)$$
$$w = s(\cos\beta + i\sin\beta)$$
$$\overline{\quad\quad\quad\quad}$$
$$z \cdot w = rs(\cos(\alpha+\beta) + i\sin(\alpha+\beta))$$



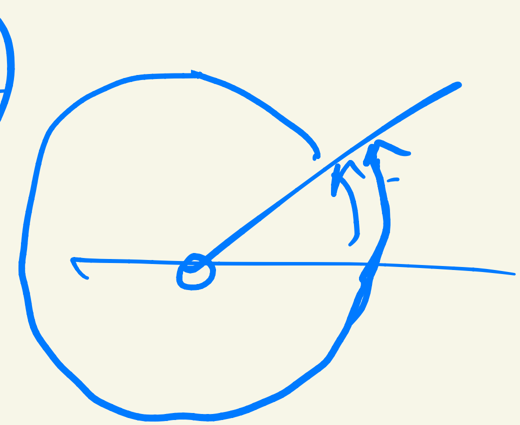
$$\alpha = \arg(z)$$
$$\beta = \arg(w)$$
$$\alpha + \beta = \arg(z \cdot w)$$

$$|z \cdot w| = r \cdot s = |z| \cdot |w|$$

If α is a valid argument for z

then

$(\forall k\in\mathbb{Z})\,(\alpha + 2k\pi$ is also —"— $)$



$z^n = 1$ solutions: n^{th} roots of unity

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solution: $1 = |z^n| = |z|^n \implies |z| = 1$

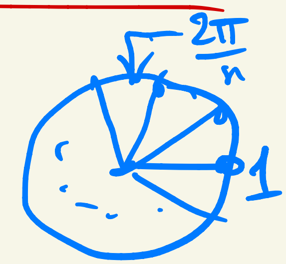
b/c $|z| \in \mathbb{R}$, $|z| \geq 0$

$\therefore\ z = \cos\theta + i\sin\theta$

$1 = z^n = \cos(n\theta) + i\sin(n\theta)$

$\therefore\ \cos(n\theta) = 1$

$\therefore\ n\theta = 2k\pi$



$\therefore\ \theta = \frac{2k\pi}{n} =$

$\implies 0, \frac{2\pi}{n}, \frac{4\pi}{n}, \ldots, \frac{2(n-1)\pi}{n}, \cancel{\frac{2n\pi}{n}} = 2\pi$

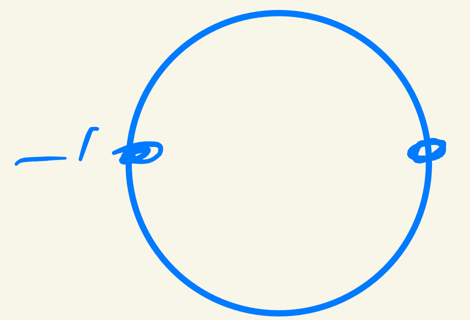
$\boxed{\text{HW}}$ The sum of the n^{th} roots of unity is 0, assuming $n \geq 2$

1^{st} root of unity : $z^1 = 1$, i.e., $\underline{z = 1}$

2^{nd} — " — : $z^2 = 1 \iff z^2 - 1 = 0$

$\iff (z+1)(z-1) = 0$

$\iff z = \begin{cases} 1 \\ -1 \end{cases}$



$n=3$

1

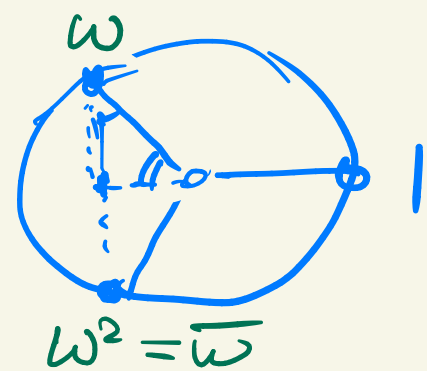
$-\frac{1}{2}+\frac{\sqrt{3}}{2}i$

$-\frac{1}{2}-\frac{\sqrt{3}}{2}i$

$z^3=1$

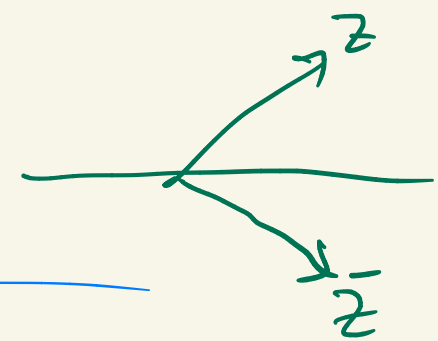
other solution

$0=z^3-1=(z-1)(z^2+z+1)$



if $z^2+z+1=0$

$z=\frac{-1\pm\sqrt{3}}{2}$



$z=a+bi$

$\bar{z}=a-bi$ "z-conjugate"

$z\cdot\bar{z}=a^2+b^2=|z|^2$ $\qquad |z|=\sqrt{z\cdot\bar{z}}$

$\overline{z+w}=\bar{z}+\bar{w}$

$\overline{z\cdot w}=\bar{z}\cdot\bar{w}$

$\frac{z\cdot\bar{z}}{|z|^2}=1 \quad \therefore \boxed{\frac{1}{z}=\frac{\bar{z}}{|z|^2}}$

L6

$a_n \in \mathbb{R}$

$\lim\limits_{n\to\infty} a_n = L \quad \in \mathbb{R}$

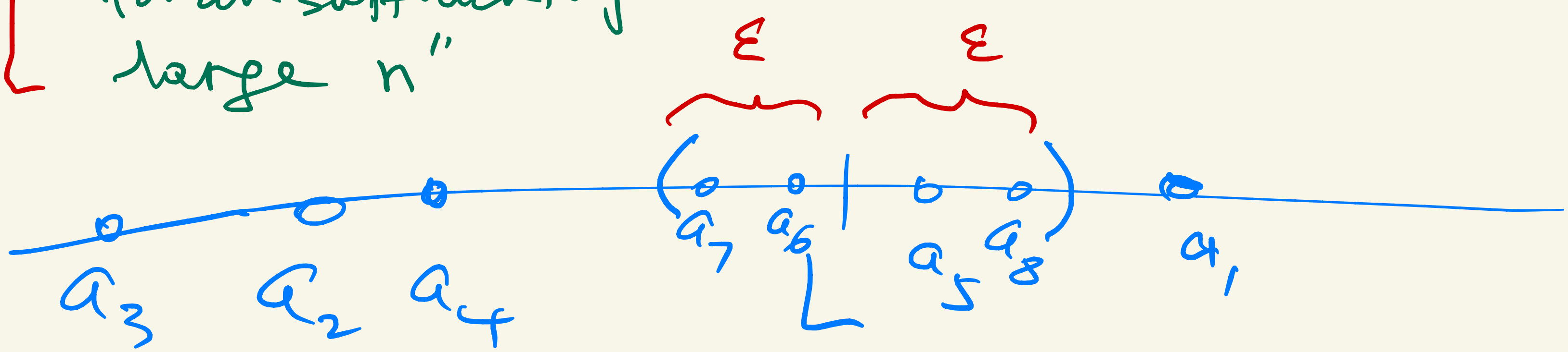
means: for every $\varepsilon > 0$

eventually $|a_n - L| < \varepsilon$

$(\forall \varepsilon > 0)\underbrace{(\exists n_0)(\forall n > n_0)}(|a_n - L| < \varepsilon)$

eventually { "for all sufficiently large n"

Quantifier game



$(\forall \varepsilon>0)(\exists n_0)(\forall n>n_0)(|d_n|<\varepsilon)$

$\lim d_n = 0$

Quantifier game for $d_n = \frac{1}{n}$

Simplest example

$d_n = 0$

$d_n = \frac{1}{n}$

simplest positive sequence

ε (Devil brings)

n_0 (we name it)

need: $(\forall n>n_0)\left(\frac{1}{n}<\varepsilon\right)$

$n_0 := \left\lceil \frac{1}{\varepsilon} \right\rceil$

$e_n = 2^{-n} \longrightarrow 0$

$\lim_{n\to\infty} 2^{-n} = 0$

given ε find n_0

s.t. $(\forall n>n_0)(2^{-n}<\varepsilon)$

$-n < \log_2 \varepsilon$

$n > \log_2 \frac{1}{\varepsilon}$

$$\boxed{n_0 := \log_2 \frac{1}{\varepsilon}}$$

$b_0 \quad b_1 \quad b_2 \quad b_3$

$1, 0, 1, 0 \ldots$

$$b_n = \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

$\lim_{n\to\infty} b_n \overset{?}{=} 0$ NOT: Devil has to win

$\varepsilon := 1$

opponent chooses n_0

Devil chooses $n := 2n_0 + 2$

Q to Judge:

$|b_n| \overset{?}{<} \varepsilon$

$n :=$ smallest even number $> n_0$

i.e. $n = \begin{cases} n_0+2 & \text{if } n_0 \text{ is even} \\ n_0+1 & \text{if } n_0 \text{ is odd} \end{cases}$