PROBLEM SESSION

2023-11-03

1

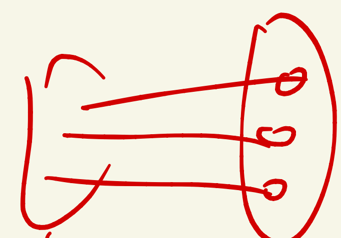
11.23 (a) $\sum_{i=1}^{\ell} x_i = n$ $\quad x_i > 0$

$$f : \underbrace{\left\{ (x_1, \dots, x_\ell) \;\middle|\; \sum_{i=1}^{\ell} x_i = n,\ x_i > 0 \right\}}_{R} \longrightarrow INCR(\ , \ )$$

$$f(t) := \sum_{i=1}^{t} x_i \qquad 1 \le t \le \ell$$

$$\boxed{INCR(k, m) = \binom{m}{k}}$$

$[k] \to [m]$



k m

f is $[\ell] \to [n]$ X

not a bijection n

$R \to INCR(?, ?)$

$f(\ell) = \underline{n}$ not surjective: not all incr. fctns $[\ell] \to [n]$ satisfy $f(\ell) = n$

11.23 (a)

$$F : \underbrace{\left\{(x_1,\dots x_\ell) \;\middle|\; \sum_{i=1}^{\ell} x_i = n,\ x_i > 0\right\}}_{R} \longrightarrow INCR(\ell-1, n-1)$$

$$f(t) := \sum_{i=1}^{t} x_i \qquad 1 \le t \le \ell - 1$$

$$\therefore\ \boxed{|R| = \binom{n-1}{\ell-1}}$$

$F : R \longrightarrow INCR(\ell-1, n-1)$ ✓ $\qquad \sum_{i=1}^{\ell-1} x_i \le n-1$

$f(0)=0$

$F(x_1 \dots x_\ell) = f \qquad F(x_1' \dots x_\ell') = f'$

F injective : $f = f' \Rightarrow (\forall i)\ (x_i = x_i')$ Pf $x_i = f(i) - f(i-1)$

$\qquad \| $

$x_i' = f'(i) - f'(i-1)$

F surjective : given $f \in INCR(\ell-1, n-1)$ we can find $(x_1 \dots x_\ell) \in R$ s.t. $F(x_1 \dots x_\ell) = f$

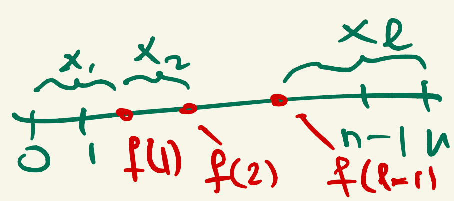
Pf: let $x_i = f(i) - f(i-1)$ $\quad [f(0) := 0,\ f(\ell) := n]$ $\quad i = 1, \dots, \ell$

NT verify: $x_i > 0$ $\quad$ yeah: $f(i-1) < f(i)$ ✓

including $f(\ell-1) < f(\ell) = n$

$\sum x_i =$

$f(1) - f(0) + f(2) - f(1) + \dots + f(\ell) - f(\ell-1) = f(\ell) - f(0) = n - 0 = n$ ✓



11.23(b) $\sum_{i=1}^{\ell} z_i = r$ (x_i, n) $z_i \geq 0$

(b1) bijection w part (a) $S(\ell, r) \to R(\ell, n)$

$n = ?$

$R(\ell, n) = \{(x_1 \cdots x_\ell) \mid x_i > 0, \sum_{i=1}^{\ell} x_i = n\}$

$\to S(\ell, r) = \{(z_1 \dots z_\ell) \mid z_i \geq 0, \sum_{i=1}^{\ell} z_i = r\}$

$x_i := z_i + 1$ so $x_i > 0$, $\sum_{i=1}^{\ell} x_i = r + \ell =: n$

bijection $S(\ell, r) \to R(\ell, r+\ell)$

$F(z_1 \dots z_\ell) = (z_1+1, \dots, z_\ell+1) \in R(\ell, r+\ell)$

claim ✓

NTS: this is a bijection

inverse $F^{-1}(x_1 \dots x_\ell) = (x_1 - 1, \dots, x_\ell - 1)$ ✓

$|R(\ell, n)| = \binom{n-1}{\ell-1}$ $\therefore$ $|S(\ell, r)| = \binom{r+\ell-1}{\ell-1}$

$|S(\ell, n)| = \binom{n+\ell-1}{\ell-1}$

11.23 (b2) $S(\ell, r) \to \text{NONDECR}(?,?) = \binom{n+\ell-1}{\ell-1}$

$(\ell-1, n+1)$

$\boxed{\text{Do}}$

11.25 $\sum_{i=1}^{h} y_i \leq q$ (a) $y_i > 0$ (b) $y_i \geq 0$

reduce to 11.23

(a) $F(y_1 \cdots y_h) = (y_1' \cdots y_{h+1}') \to R(h+1, q+1)$

$R(\ell, n)$
? ?

$y_i' = y_i$ for $i \leq h$

$y_{h+1}' = \underline{q} - \sum_{i=1}^{h} y_i$

wrong: $n = q$ does not work

instead: $n := q+1$

$y_{h+1}' := (q+1) - \sum_{i=1}^{h} y_i$

~~$\#\text{sol's}: \binom{(q+1)+(h+1)-1}{h+1} = \binom{q+h+1}{h+1}$~~

~~$\text{NONDECR}(k, m) = \binom{m+k-1}{k}$~~

$|R(\underbrace{h+1}_{\ell}, \underbrace{q+1}_{n})| = \binom{q}{h}$

$|R(\ell, n)| = \binom{n-1}{\ell-1}$

11.25 (b)

$$\sum_{i=1}^{k} y_i \le q \quad , \quad y_i \ge 0 \quad \xrightarrow{\text{bij.}} S(\ell, r)$$

$$y'_{k+1} = q - \sum_{i=1}^{k} y'_i$$

$\ell \leftarrow k+1$, $r \leftarrow q$

$$|S(\ell,r)| = \binom{\ell+r-1}{\ell-1}$$

$$\left| \left\{ (y_1, \dots, y_k) \;\middle|\; \sum_{i=1}^{k} y_i \le q \, , \; y_i \ge 0 \right\} \right| = \underline{\underline{\binom{k+q}{k}}}$$

bijection $\updownarrow$

$$S(k+1, r)$$

11.27

$$H(k,m) := \left\{ h : [k] \to [m] \;\middle|\; (\forall i)\big(h(i+1) \geq h(i)+2\big) \right\} \xrightarrow{\text{bij.}} \mathrm{INCR}(k, \overset{m-k+1}{?})$$

- $f(i) := h(i) - i + 1 \qquad i = 1 \ldots k$

$f(1) = h(1)$

$f(2) = h(2) - 1$

$f(3) = h(3) - 2$

$\vdots$

so $\quad 1 \leq f(i) \leq m-k+1$

$f(i+1) \geq f(i) + 1 \quad$ b/c

$$\begin{aligned} f(i+1) &= h(i+1) - i \\ f(i) &= h(i) - i + 1 \end{aligned}$$

$$f(i+1) - f(i) = h(i+1) - h(i) - 1 \geq 2 - 1 = 1 \quad \checkmark$$

$F(h) = f \qquad F^{-1}(f) = h$ defined by $h(i) = f(i) + i - 1$

$\mathrm{INCR}(k, q) = \binom{q}{k} \qquad \therefore \; |H(k,m)| = \binom{m-k+1}{k}$

11.61 $\forall$ tree is bipartite, i.e., 2-colorable

Sol 1

Pf induction on n

Base $n=1$ $m=0$ •

Now assume $n \geq 2$

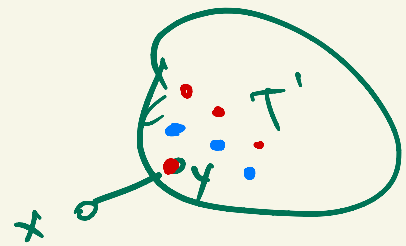
I.H. same true for all $n' \leq n-1$

all trees w $\leq n-1$ vertices are bipartite

NTS Given a tree T of order n, T is "

T has a pendant vx x

$T' = T - x$



T' 2-colorable by IH

pick a 2-coloring of T'

color x with the color $\neq$ color(y)

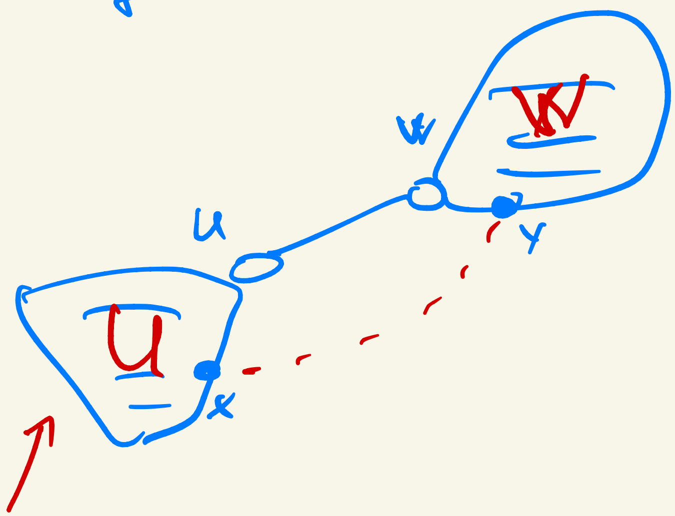
$\therefore$ 2-coloring of T' extends to 2-coloring of T

2nd sol'n:

IH as before

inductive step: input: tree T of order n

pick any edge $\{u,w\}$



U, W induce subtrees

1. no cycle
2. u reaches all of U

so $T[U]$ conn.

same for W

U all vertices accessible from u w/o passing through w

W " acc. from w w/o passing through u

IH $\Rightarrow$ $\left.\begin{array}{l} T[U] \\ T[W] \end{array}\right\}$ bipartite

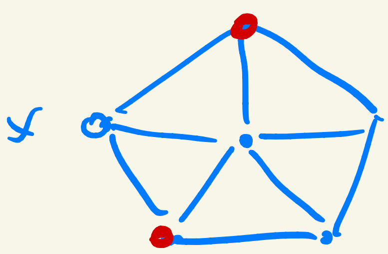
fix a red/blue coloring of each

if $col(u) \neq col(w) \Rightarrow$ we colored T

o/w swap the colors of W

11.69

find graph G s.t. $G \not\supseteq K_4$ ·
$\chi(G) = 4$ ·



Pf $G \not\supseteq K_4$

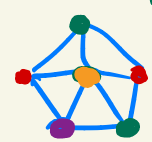
Pf by contradiction: assume $\exists\ K_4$ sub-graph

WLOG $x \in V(K_4)$ "by symmetry"

$\therefore$ all its neighbors must belong to this K_4

but two of the neighbors are not adjacent $\rightarrow\leftarrow$

$\chi(G) \leq 4$ easy



$\chi(G) \geq 4$: NTS G is NOT 3-colorable

Pf by contradiction: pick a 3-coloring

11.59: $\chi(C_5) = 3$ $\therefore$ outer C_5 uses all the 3 colors

$\therefore$ NO LEGAL OPTION REMAINS FOR CENTER VERTEX $\rightarrow\leftarrow$

11.71 $\chi(G) \le 1+\Delta$ $\qquad$ Δ = max degree

Pf induction on n

$n=1$ $\quad \Delta = 0$ $\qquad \bullet K_1 \quad \chi(K_1)=1$

Now $n \ge 2$

IH thm true for all graphs with $\le n-1$ vertices

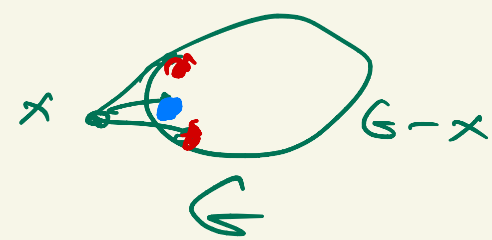
ind. step: we a given G w n vertices

pick any $x \in V(G)$

$\Delta(G-x) \le \Delta(G) = \Delta$ (degrees don't increase)

$$\therefore \chi(G-x) \underset{\text{IH}}{\le} 1+\Delta(G-x) \le 1+\Delta$$

Claim: any $1+\Delta$-coloring of $G-x$ extends to a $1+\Delta$ " of G



fix Δ-coloring of $G-x$

need to color x, avoiding colors of its neighbors

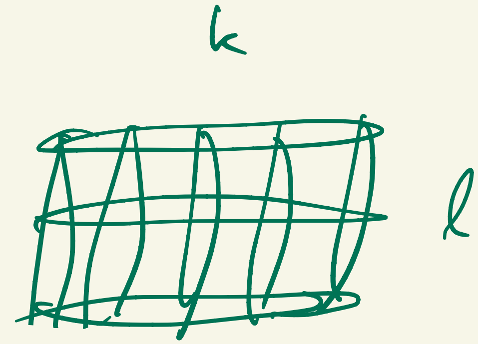
#neighbors $\le \Delta$ we have $\Delta+1$ colors $\Rightarrow$ extra color exists

not used by neighbors

11.85

$$\chi(k\times \ell \text{ rook graph}) = \max(k,\ell)$$

WLOG $k \ge \ell$ $\qquad G = K_k \,\square\, K_\ell$



(1) $\chi(G) \ge k$

Pf $\quad G \supseteq K_k$ ✓

(2) $\chi(G) \le k$ $\qquad V = \{0,\dots,\ell-1\}\times\{0,\dots,k-1\}$

We need to color G by k colors

$(0,0)$	$(0,1)$	$(0,2)$		$(0,k-1)$
$(1,0)$	$(1,1)$	$(1,2)$	$(1,3)\dots$	$(1,k-1)$
	$(2,1)$	$(2,2)$	$(2,3)\dots$	$(2,k-1)$

0	1	2	3	$\dots$	$k-1$
1	2	3	4	$\dots$	0
2	3	4	5	$\dots$	1
	$\vdots$				
$\ell-1$	ℓ	$\ell+1$	$\dots$		$\ell-2$

$$\text{color}(i,j) = (i+j \mod k)$$

This is a legal coloring:

assume $\text{col}(i,j_1) = \text{col}(i,j_2)$ $\therefore$ $i+j_1 \equiv i+j_2 \pmod{k}$ $\therefore$ $j_1 \equiv j_2 \pmod{k}$

but $0 \le j_1, j_2 \le k-1 \Rightarrow j_1 = j_2$

Similarly if $\text{col}(i_1,j) = \text{col}(i_2,j)$ $\dots$ $0 \le i_1, i_2 \le \ell-1 \le k-1 \Rightarrow i_1 = i_2$