sequence of numbers

DEF $a_0, a_1, a_2, \ldots$ is **bounded**

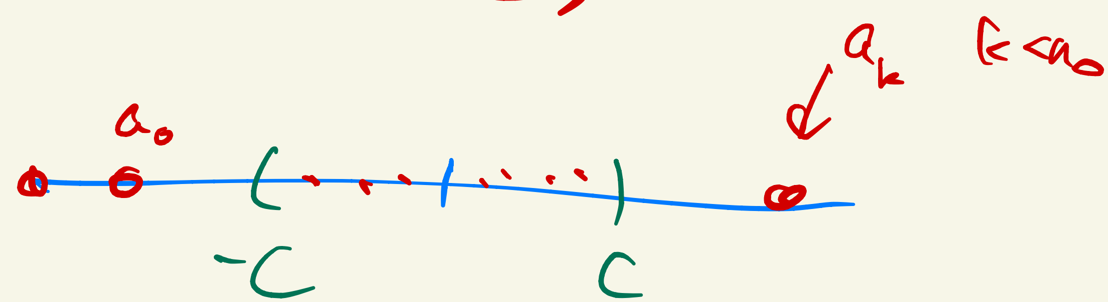
if $(\exists n_0, C)(\forall n)(n \geq n_0 \Rightarrow |a_n| \leq C)$ $(*)$

$(\exists C)(\forall n)(|a_n| \leq C)$ $(**)$

$(\exists D)(\forall n)(|a_n| \leq D)$

$(**) \Rightarrow (*)$ ✓

$(*) \overset{?}{\Rightarrow} (**)$



YES: $D := \max(|a_i| : i \leq n_0, C)$

But NO if a finite number of a_i are undefined

Asymptotic equality of sequences

$a_n \sim b_n$

if $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$

relative error if we replace b_n by a_n

$$\frac{a_n - b_n}{b_n} = \frac{a_n}{b_n} - 1 \to 0$$

$$\frac{a_n}{b_n} \to 1$$

$$\cancel{\lim_{n\to\infty} \frac{a_n}{b_n} \to 1}$$

$$3n^2 - 7n + 1000 \sim 3n^2$$

Pf
$$\frac{3n^2 - 7n + 1000}{3n^2} = \underbrace{1 - \frac{7}{3}\cdot \underset{\downarrow 0}{\frac{1}{n}}}_{\downarrow 0} + \underbrace{\frac{1000}{3}\cdot \underset{\downarrow 0}{\frac{1}{n^2}}}_{\downarrow 0} \to 1 - 0 + 0 = \underline{\underline{1}}$$

RSA cryptosystem < e-commerce

pick p, q: large random primes

n digits

prime counting fctn

$$\pi(x) := \#\{\text{primes} \leq x\}$$

$\pi(5) = 3$ 2, 3, 5

$\pi(10) = 4$ — " — , 7

$\pi(100) = 25$

PRIME NUMBER THEOREM 1896

$$\boxed{\pi(x) \sim \frac{x}{\ln x}}$$

FINDING RANDOM n-digit prime numbers:

pick random n-digit number r | $\nwarrow$ we permit initial zeros like 0017

efficient $\rightarrow$ check whether r is prime

primality checking algorithms exist – based on Fermat's little Theorem

repeat until prime found

IS THIS METHOD EFFICIENT?

$$P(r \text{ is prime}) = \frac{\pi(10^n)}{10^n} \sim \frac{10^n / \ln(10^n)}{10^n} = \frac{1}{\ln 10^n} = \frac{1}{\ln 10} \cdot \frac{1}{n}$$

one out of $\sim \underbrace{\ln 10}_{\approx 2.2} \cdot n$ n-digit numbers are prime

EFFICIENT ✓

S: set of all sequences of real numbers

is $\sim$ reflexive on S ?

$$\frac{a_n}{a_n} \overset{?}{\longrightarrow} 1$$

NO if $a_n = 0$ infinitely often

$\boxed{HW}$ If (a_n) is not event. nonzero then $(\forall b_n)(a_n \not\sim b_n)$

$S^* = \{$ eventually non-zero sequences $\}$

$n > n_0 \implies \dfrac{a_n}{a_n} = 1 \implies \lim_{n\to\infty} \dfrac{a_n}{a_n} = 1$ ✓

$\boxed{HW}$ $\sim$ is an equiv rel. on S^*

$\boxed{EX}$ $\left.\begin{array}{l} a_n \sim b_n \\ c_n \sim d_n \end{array}\right\} \implies a_n c_n \sim b_n d_n$

$$\frac{a_n}{c_n} \sim \frac{b_n}{d_n}$$

$\Downarrow$?

$a_n + c_n \sim b_n + d_n$

$\boxed{EX}$ Find a_n, b_n, c_n, d_n s.t.

$a_n \sim b_n$

$c_n \sim d_n$

$(\forall n)\ (a_n + c_n \neq 0)$

$(\forall n)\ (b_n + d_n \neq 0)$

AND $\quad a_n + c_n \not\sim b_n + d_n$

XC. If $a_n, b_n, c_n, d_n > 0$

and $a_n \sim b_n$
$c_n \sim d_n$

then $a_n + c_n \sim b_n + d_n$

$\boxed{\text{DO}}$ If $a_n \sim b_n$ then $a_n^5 \sim b_n^5$

$\boxed{\text{EX}}$ If $a_n \sim b_n$ and $a_n, b_n > 0$

then $\sqrt{a_n} \sim \sqrt{b_n}$

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then $a_n^{1/10} \sim b_n^{1/10}$

$\boxed{\text{EX}}$ $\left.\begin{matrix} a_n \sim b_n \\ a_n > 1,\ b_n > 1 \end{matrix}\right\} \not\Rightarrow \ln a_n \sim \ln b_n$?

$\boxed{\text{EX}}$

$$\left.\begin{aligned} &a_n \sim b_n \\ &a_n > 1.01 \\ &b_n > 1.01 \end{aligned}\right\} \Longrightarrow \ln a_n \sim \ln b_n$$

If $\exists\, \varepsilon > 0$

$a_n > 1 + \varepsilon$ and $a_n \sim b_n \Longrightarrow \ln a_n \sim \ln b_n$

$[b_n > 1 + \varepsilon]$

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a_n is bounded away from 1

EX If $a_n, b_n > 1$ and $a_n \sim b_n \Longrightarrow b_n$ is also bdded away from 1

a_n bounded away from 1