PRIME NUMBER THEOREM

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$$\pi(x) \sim \frac{x}{\ln x}$$

$$n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$$ Stirling's formula

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little-oh relation

$$a_n = o(b_n) \quad \text{if} \quad \lim_{n\to\infty} \frac{a_n}{b_n} = 0$$

$\boxed{\text{Do}}$

$$a_n \sim b_n \iff a_n - b_n = o(b_n)$$

$$\left.\begin{array}{l} a_n = o(c_n) \\ b_n = o(c_n) \end{array}\right\} \Longrightarrow a_n + b_n = o(c_n)$$

$$a_n = o(b_n) \Longrightarrow (\forall c)(c a_n = o(b_n))$$

little-omega

$a_n = \omega(b_n)$ if $b_n = o(a_n)$

$\boxed{\text{DO}}$ $\left.\begin{array}{l} a_n = \omega(b_n) \\ a_n = \omega(c_n) \end{array}\right\} \Longrightarrow a_n = \omega(b_n + c_n)$

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f, g polynomials, $\deg(f) < \deg(g)$

$\Longrightarrow f(n) = o(g(n))$

EXAMPLE $1000n^2 = o(n^3 - 10n^2 - 10^6)$

$n^3 - 10n^2 - 10^6 = \omega(1000n^2)$

$\boxed{\text{HW}}$ $a_n = \omega(b_n) \Longrightarrow$ for all suff. large n, $|a_n| > |b_n|$

$$\ln x = o(x)$$

i.e. $$\lim_{x\to\infty} \frac{\ln x}{x} = 0$$ b/c $$\frac{1/x}{1} = \frac{1}{x} \to 0$$

apply L'Hôpital's Rule

also spelled L'Hospital

If $\begin{cases} f(x) \to \infty, & \exists f' \\ g(x) \to \infty, & \exists g' \end{cases}$ OR $\begin{matrix} f(x) \to 0 \\ g(x) \to 0 \end{matrix}$

and $\exists \lim\limits_{x\to\infty} \dfrac{f'(x)}{g'(x)}$

then $\exists \lim\limits_{x\to\infty} \dfrac{f(x)}{g(x)} = \lim\limits_{x\to\infty} \dfrac{f'(x)}{g'(x)}$

$a_n = 2n^2 \qquad b_n = 2n^2 + n$

$$\frac{a_n}{b_n} \longrightarrow 1 \qquad a_n \sim b_n$$

$$\underbrace{a_n - b_n}_{-n} = o(a_n)$$

$2n^2$

$$\lim \frac{\ln x}{\sqrt{x}} = 0$$

L'Hôpital

$$\frac{1/x}{\frac{1}{2\sqrt{x}}} = \frac{2\sqrt{x}}{x} = \frac{2}{\sqrt{x}} \longrightarrow 0$$

$$\therefore \ln x = o(\sqrt{x})$$

$\underline{\forall \varepsilon > 0}$ $\quad \ln x = o(x^{\varepsilon})$

$$\frac{\ln x}{x^{\varepsilon}} \to 0$$

$$\frac{1/x}{\varepsilon x^{\varepsilon - 1}} = \frac{1}{\varepsilon} \cdot \frac{x^{1-\varepsilon}}{x} = \frac{1}{\varepsilon} \cdot \frac{1}{x^{\varepsilon}} \downarrow 0$$

EXAMPLE

$\varepsilon = 0.01$

$$x^{\varepsilon} = x^{0.01} = \sqrt[100]{x}$$

$$\ln x = o(\sqrt[100]{x})$$

$$\ln x = o\left(x^{\varepsilon}\right)$$

2^{nd} proof: without L'Hôpital

$$y := x^{\varepsilon}$$

$$\ln y = \varepsilon \ln x$$

$$\frac{\ln x}{x^{\varepsilon}} = \frac{\frac{1}{\varepsilon}\ln y}{y} = \frac{1}{\varepsilon}\cdot\frac{\ln y}{y} \longrightarrow 0$$

if $x \to \infty$ then $y \to \infty$

"bootstrapping"

For $C > 1$, $\quad n = o(C^n)$

Pf $\quad x = o(C^x)$

$y := C^x$

$\ln y = x \ln C \qquad \ln C > 0$

$$\frac{x}{C^x} = \underbrace{\frac{1}{\ln C}}_{\text{const}} \cdot \underbrace{\frac{\ln y}{y}}_{\downarrow \, 0} \longrightarrow 0$$

$$(\forall c)(\forall C > 1)\left(n^c = o(C^n)\right)$$

Proof

$$x^c = o(C^x)$$

$$\frac{x^c}{C^x} = \left(\frac{x}{C^{x/c}}\right)^c = \left(c \cdot \frac{x/c}{C^{x/c}}\right)^c = \left(c \cdot \frac{y}{C^y}\right)^c \to 0$$

$$y = x/c$$

$\downarrow$

0

bootstrapping

$$n^c = o\left(C^{\sqrt{n}}\right)$$

$$x^c = o\left(C^{\sqrt{x}}\right)$$

Pf $\quad y := \sqrt{x} \qquad x = y^2 \qquad x^c = y^{2c}$

$$y^{2c} \overset{?}{=} o\left(C^y\right)$$

by previous result

— bootstrapping

EX.

$\forall c$

$\forall C > 1$

$\forall \varepsilon > 0$

THM $\quad x^c = o\left(C^{x^\varepsilon}\right)$

"exponential growth beats polynomial growth"

EXAMPLE: $x^{100} = o\left(1.0001^{\sqrt[127]{x}}\right)$

$\boxed{\text{DO}}$ $\left.\begin{aligned} a_n &= o(b_n) \\ b_n &= o(c_n) \end{aligned}\right\} \Longrightarrow a_n = o(c_n)$

$\boxed{\text{DO}}$ If $a_n = o(b_n)$ then b_n is eventually nonzero

$$\frac{a_n}{b_n} \to 0$$

Find $a_n, b_n \longrightarrow 0$ and $a_n = o(b_n)$

Sol: $a_n = \frac{1}{n^2}$ $\quad b_n = \frac{1}{n}$ $\quad \frac{a_n}{b_n} = \frac{1}{n} \to 0$

$\boxed{\text{DO}}$ $\forall$ sequence (a_n) $\exists$ seq. (b_n) s.t. $a_n = o(b_n)$

HW

(a)
$$\left.\begin{array}{l} a_n = o(b_n) \\ a_n = o(c_n) \\ (\forall n)(b_n + c_n \neq 0) \end{array}\right\} \not\Longrightarrow a_n = o(b_n + c_n)$$

(b)
$$\left.\begin{array}{l} a_n = o(b_n) \\ a_n = o(c_n) \\ b_n, c_n > 0 \end{array}\right\} \Longrightarrow \text{—\,''\,—}$$

big-Oh notation

$$a_n = O(b_n)$$ "a_n is big-Oh of b_n"

if $(\exists C)$(for all suff. large n)$(|a_n| \leq C|b_n|)$

EXAMPLE $1000n^3 = O(n^3 - 100n - 10^6)$

implied constant: any $C > 1000$

$\boxed{\text{HW}}$ If $a_n = o(b_n)$ then $a_n = O(b_n)$

If $a_n \sim b_n$ —"— —"—

big-Omega notation

$$a_n = \Omega(b_n) \quad \text{if} \quad b_n = O(a_n)$$

big-Theta

$$\underbrace{a_n = \Theta(b_n)}_{\text{same rate of growth}} \quad \text{if} \quad \begin{cases} a_n = O(b_n) \\ \text{and} \\ a_n = \Omega(b_n) \end{cases}$$

$\boxed{\text{HW}}$ Θ is an equiv. rel. on all sequences