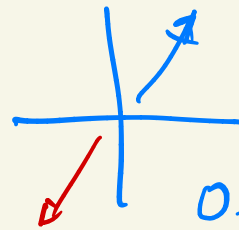
PROBLEM SESSION | 2023-11-10

12.21 $z = 5 + 12i$ $|z| = \sqrt{5^2 + 12^2} = 13$

$\arg(z) = \tan^{-1}\left(\frac{12}{5}\right) \approx 1.176$ ←



$0 \leq 1.176 \leq \frac{\pi}{2}$

$\arg(-z) \neq \tan^{-1}\left(\frac{-12}{-5}\right)$

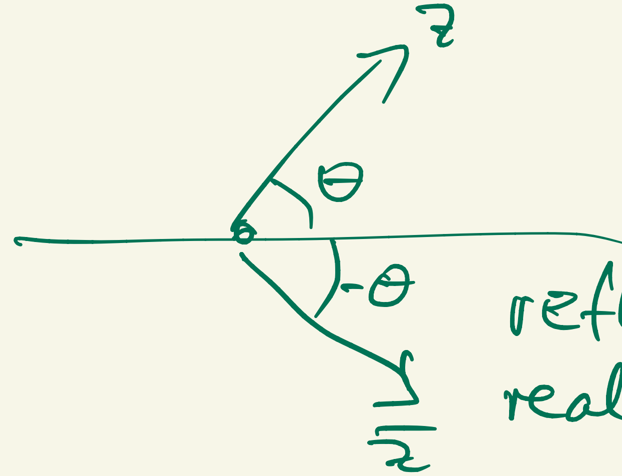
$\tan^{-1}\left(\frac{12}{5}\right) + \pi$

all arguments of z: $\tan^{-1}\left(\frac{12}{5}\right) + 2k\pi$

all arguments of $-z$: $\tan^{-1}\left(\frac{12}{5}\right) + (2k+1)\pi$

NEED TO CHECK QUARTILE

12.27 $\arg(\bar{z}) = 2\pi - \arg(z)$



reflection in real axis

12.55 If $k \mid l$
and z is a k^{th} root of unity
then — " — an l^{th} — " —

Assn. $z^k = 1$

DC. $z^l = 1$

Pf let $l = km$ $(\exists m \in \mathbb{Z})$

$$z^l = z^{km} = (z^k)^m = 1^m = 1 \checkmark$$

12.59 If 3^{rd} roots of unity are $\omega_0, \omega_1, \omega_2$
then 6^{th} " " $\pm\omega_0, \pm\omega_1, \pm\omega_2$

If k is odd then the $(2k)^{th}$ roots of unity
are $\pm$ k^{th} roots of unity

Pf (1) Let z be a k^{th} root of unity.
Claim z and $-z$ are $(2k)^{th}$ -"-

Pf Assn: $z^k = 1$
DC: $(\pm z)^{2k} = 1$

Pf $(\pm z)^{2k} = \underbrace{(\pm 1)^{2k}}_{1} \cdot z^{2k} = (z^k)^2 = 1^2 = 1$ ✓

(2) If w is a $(2k)^{th}$ root of unity then w or $-w$
is a k^{th} root of unity

Pf Assn $w^{2k} = 1$ $\longrightarrow$ $0 = (w^k)^2 - 1 = (w^k - 1)(w^k + 1) = 0$
DC $w^k = 1 \vee (-w)^k = 1$ $\quad$ $\underline{w}^k = \pm \underline{1}$ but $\underline{(-w)}^k = \underbrace{(-1)^k}_{ODD} \cdot w^k = -w^k = 1$
$w^k = 1$

12.63 n^{th} roots of unity: $\underset{\overset{\shortparallel}{1}}{\omega_0}, \omega_1, \ldots, \omega_{n-1}$

$S := \sum_{j=0}^{n-1} \omega_j$ Claim $S = 0$

Pf Lemma If u, v are n^{th} roots of unity then $u \cdot v$ n^{th} root of unity

Pf $(uv)^n = u^n v^n = 1 \cdot 1 = 1$ ✓

If $|z| = 1$ then $z^{-1} = \bar{z}$

Pf $z \cdot \bar{z} = |z|^2 = 1$ ✓

$R = \{\omega_0, \omega_1, \ldots, \omega_{n-1}\}$

$f : x \mapsto x \cdot \omega_1$ $f(x) = x \cdot \omega_1$

Claim $f : R \to R$ bijection

indeed $g(x) = x \cdot \omega_1^{-1}$

Remark: $\omega_1^{-1} = \bar{\omega}_1$

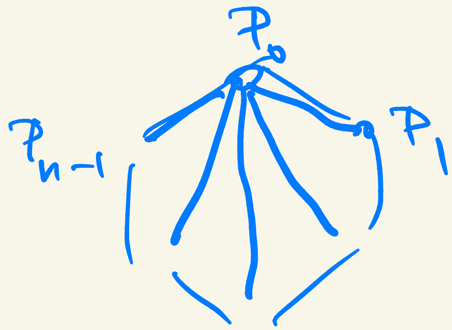
$\therefore$ Sum of R is the same as the sum of $R \cdot \omega_1 = \{\omega_i \omega_1 \mid i = 0, \ldots, n-1\}$

$\therefore$ $S = S \cdot \omega_1$

$\therefore$ $S(\underbrace{\omega_1 - 1}_{\text{not } 0}) = 0$ $\therefore$ $S = 0$

↑ real axis

12.69xc



inscribed in the unit circle

$$\prod_{j=1}^{n-1} \overline{P_0P_j} = n$$

Pf P_j represented by w_j $\qquad w_0, \dots, w_{n-1}$: n^{th} roots of unity

$w_0 = 1$

NTS $f(1) = \prod_{j=1}^{n-1} |1 - w_j| = n$

Pf $\quad z^n - 1 = \prod_{j=0}^{n-1} (z - w_j) = (z-1) \underline{\prod_{j=1}^{n-1} (z - w_j)}$

$$g(z) := \prod_{j=1}^{n} (z - w_j) = \frac{z^n - 1}{z - 1} = z^{n-1} + z^{n-2} + \dots + z + 1 =: f(z)$$

$z \neq 1$

$f(1) = 1 + \dots + 1 = n$ ✓

We proved $(\forall z \neq 1)(g(z) = f(z))$

$\therefore$ by continuity $g(1) = f(1)$ ✓

12.85 Define $a_n \to -\infty$

$$(\forall L)\underbrace{(\exists n_0)(\forall n)(n > n_0}_{\text{for all sufficiently large } n} \Longrightarrow a_n < L)$$

$\cancel{\forall L < 0}$

13.17 (a) Define "a_n eventually increasing"

$$(\exists n_0)(\forall n)(n > n_0 \Longrightarrow a_{n+1} > a_n)$$

(b) Find seq. $b_n \to \infty$ but b_n is not event. increasing

$1 \quad 0 \quad 3 \quad 2 \quad 5 \quad 4 \quad 7 \quad 6 \quad \ldots$

$b_n = 2n + (-1)^n$ or something like it

13.33 (a) if a_n convergent then a_n is bounded

DEF (a_n) is bounded if

$$(\exists C)(\forall n)(|a_n| < C) \quad *$$

$$(\exists C, n_0)(\forall n)(n > n_0 \Rightarrow |a_n| < C) \quad **$$

ASSN $a_n \to L$ $\quad L \in \mathbb{R}$

$$(\forall \varepsilon)(\exists n_0)(\forall n)(n > n_0 \Rightarrow |a_n - L| < \varepsilon)$$

$$L - \varepsilon < a_n < L + \varepsilon$$

$\varepsilon := 1$

need to assign specific value to ε

this assigns a value to n_0

$$C := \max\{|L-1|, |L+1|\}$$

$$\Rightarrow (n > n_0 \Rightarrow |a_n| < C)$$

If using DEF *

$$C \leftarrow \max\{C, |a_0|, \ldots, |a_{n_0}|\} + 1$$

13.35 if: $a_n \to -\infty$ then a_n is bdded from above

Pf Assn: $(\forall L)(\exists n_0)(\forall n)(n > n_0 \Rightarrow a_n < L)$

DC: $(\exists C, n_1)(\forall n)(n > n_1 \Rightarrow a_n < C)$ **

n_0 depends on L

Set $L := 0$

this choice defines n_0

now let $C := 0$

$n_1 := n_0$

If DEF * used

then

$C := \max\{a_j \mid j \leq n_0\} + 1$

(9)

13.41 $\{z^n\}$

Claim (a) If $|z|>1$ then z^n diverges

(b) If $|z|<1$ then z^n converges

13.43 (c) If $|z|=1$ then when does (z^n) converge?

Claim (a), (c) If $|z|\geq 1$ then z^n converges

$\Longleftrightarrow$ $\underline{z=1}$

Pf $\Leftarrow$ ✓

$\Longrightarrow$ $\underline{\text{ASSN}}$ $\exists$ limit, call it $L = \lim_{n\to\infty} z^n = \lim_{n\to\infty} z^{n+1}$

$\therefore \lim_{n\to\infty} (z^{n+1} - z^n) = 0$

$z^n \cdot (z-1) \longrightarrow 0$

$0 \leq |z-1| \leq |z^n||z-1| = |z^n(z-1)| \longrightarrow 0$

by squeeze principle $|z-1| = 0$

$\therefore z-1=0$

$z=1$ ✓

13.41 (5) if $|z|<1$ then Claim $z^n \to 0$

b/c $|z^n - 0| = |z^n| = |z|^n \to 0$ ✓

$\boxed{\text{DO}}$ Lemma $r \in \mathbb{R}$, $0 \le r < 1$ then $r^n \to 0$

13.87 For what set $S \subseteq \mathbb{R}$ do we have

$$\sup(S) < \inf(S) \quad ?$$

Claim: $\iff S = \emptyset$

$\Rightarrow$ NTS $S \neq \emptyset \Rightarrow \sup(S) \ge \inf(S)$

Pf pick $x \in S$ but then $\inf(S) \le x \le \sup(S)$ ✓

$\Leftarrow$

DEF $\sup(S) \in \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$

$\sup(\emptyset) = ?$ $\forall$ real is an upper bound on $\emptyset$ $\therefore$ $\sup(\emptyset) = -\infty$

DEF t is an upper bound for S if $(\forall x \in S)(x \le t)$

$\inf(\emptyset) = \infty$

13.89 $\limsup$ / $\liminf$ $(-1)^n\left(1+\frac{1}{n}\right)$ $\begin{cases} 1 \\ -1 \end{cases}$

13.103 $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}$ $n \geq 1$

13.91 find a_n, b_n s.t. $\limsup (a_n + b_n) < \limsup a_n + \limsup b_n$

always $\leq$

Soln: $a_n = 0\ 1\ 0\ 1\ \cdots$ $\limsup = 1$

$b_n = 1\ 0\ 1\ 0\ \cdots$ $\limsup = 1$

$a_n + b_n = 1\ 1\ 1\ 1\ \cdots$ $1 < 1 + 1$

13.29 Assume $(\exists r)\left(\frac{F_{n+1}}{F_n} \to r\right)$ Find r

Pf $r = \lim \frac{F_{n+1}}{F_n} = \lim \frac{F_{n+2}}{F_{n+1}} = \lim \frac{F_n + F_{n+1}}{F_{n+1}} = \lim\left(1 + \frac{F_n}{F_{n+1}}\right) = 1 + \frac{1}{r}$

$r = 1 + \frac{1}{r}$

$r^2 - r - 1 = 0$

$r = \frac{1 \pm \sqrt{5}}{2}$ $\begin{cases} \checkmark\ \frac{1+\sqrt{5}}{2} \text{ golden ratio} \\ < 0, \text{ but } F_n \geq 0 \end{cases}$

12

13.27 $\left.\begin{array}{l} a_n \to L \\ a_n \to M \end{array}\right\} \Rightarrow L = M$

$(\forall \varepsilon > 0)\,(\exists n_0)\,(n > n_0 \Rightarrow |a_n - L| < \varepsilon)$

$(\forall \varepsilon > 0)\,(\exists n_1)\,(n > n_1 \Rightarrow |a_n - M| < \varepsilon)$

choose $\varepsilon := \frac{|L-M|}{2}$

Δ-ineq

$n_2 := \max(n_0, n_1)$

then $n > n_2 \Rightarrow \underline{2\varepsilon} = |L-M| = |(L-a_n) + (a_n - M)| \leq$

$|L-a_n| + |a_n - M| \underline{\underline{<}} \varepsilon + \varepsilon = \underline{\underline{2\varepsilon}}$ $\rightarrow\leftarrow$

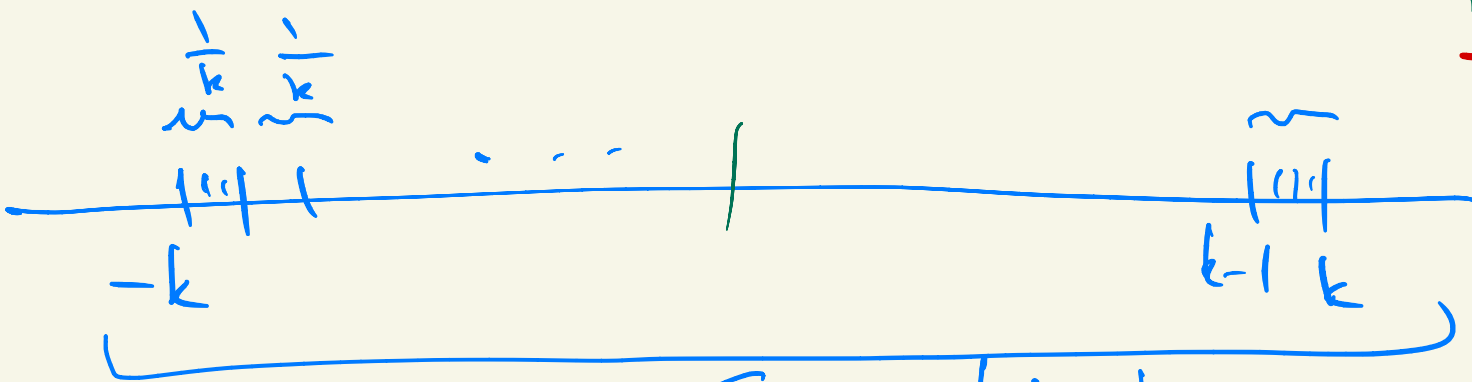
INCOMPLETE

missing cases when $a_n \to \pm\infty$ $\boxed{\text{DO}}$

13.24XC Find (a_n) s.t. $(\forall r)(\exists I \subseteq \mathbb{N}_0)(\lim_{n \in I} a_n = r)$

every real number is a limit of a subsequence

$k \in \mathbb{N}$



S_k finite sequence

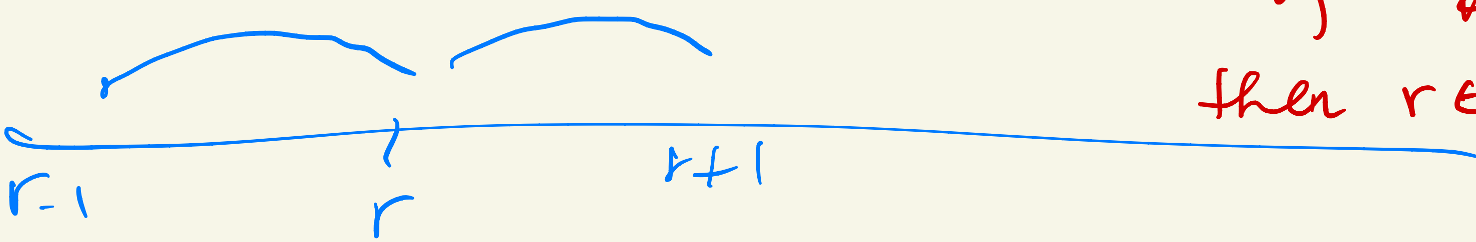
$2k^2+1$ terms

Concatenate: $S_1 \; S_2 \; S_3 \; \ldots$

Claim: this sequence works

if $k_i \geq \lceil |r| \rceil$

then $r \in [-k_i, k_i]$



13.53 XC $\quad (a_{ij})_{i,j \in \mathbb{N}_0}$

$$
\begin{array}{cccc}
a_{00} & a_{01} & a_{02} & \cdots \longrightarrow 0 \\
a_{10} & a_{11} & a_{12} & \cdots \\
a_{20} & a_{21} & a_{22} & \cdots \\
 & \downarrow & & \\
 & \infty & &
\end{array}
$$

$$a_{ij} = \frac{i}{j+1}$$

$\lim\limits_{i \to \infty} a_{ij} = \infty \qquad j^{\text{th}}$ column

$\lim\limits_{j \to \infty} a_{ij} = 0 \qquad i^{\text{th}}$ row

13.61 $s_0 = 0$

$$s_{n+1} = \sqrt{2}^{\,s_n}$$

Claim $\lim s_n = 2$

Pf LEMMA 1 $s_n < s_{n+1}$

Pf by induction: $n=0$ $s_0 = 0$ $s_1 = 1$ $s_0 < s_1$ ✓

now assume $n \geq 1$

IH true for all $n' < n$

in particular, true for $n' = n-1$: $s_{n-1} < s_n$

$$\therefore \quad \underbrace{\sqrt{2}^{\,s_{n-1}}}_{= s_n} < \underbrace{\sqrt{2}^{\,s_n}}_{= s_{n+1}} \quad ✓$$

LEMMA 2 $s_n < 2$

Pf by induction. $s_0 = 0 < 2$

IH $s_n < 2$

DC $s_{n+1} < 2$

$$s_{n+1} = \sqrt{2}^{\,s_n} \overset{\text{IH}}{<} \sqrt{2}^{\,2} = 2 \quad ✓$$

So $s_n \nearrow$, < 2

$\Rightarrow \exists L \le 2$ s.t. $\lim s_n = L$

LEMMA 3 $L = 2$

$$L = \lim s_n = \lim s_{n+1} = \lim \sqrt{2}^{s_n} = \sqrt{2}^{L}$$

↑ continuity of 2^x fctn

$\therefore L = \sqrt{2}^{L}$

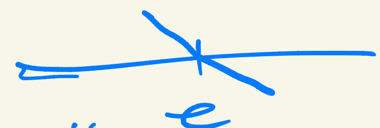
$\boxed{\sqrt{2} = L^{1/L}}$ look at $x^{\frac{1}{x}} =: f(x)$ $x > 1$

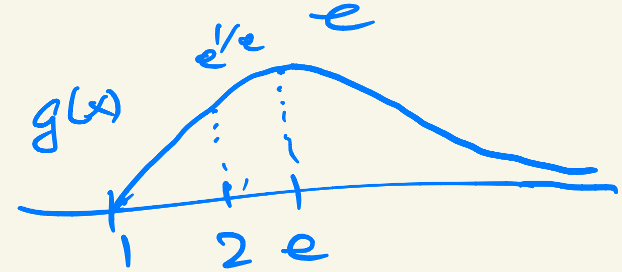
$L=2$ is a solution

NTS: there is no sol'n $L' < 2$

$$g(x) = \ln f(x) = \frac{\ln x}{x}$$

$$g'(x) = \frac{\frac{1}{x}\cdot x - \ln x}{x^2} = \frac{1 - \ln x}{x^2}$$





$x = 2 < e$

$\therefore x^{1/x} = \sqrt{2}$ has only 1 sol'n $x < e$

this sol is $x = 2$ $\therefore$ $L = 2$ ✓