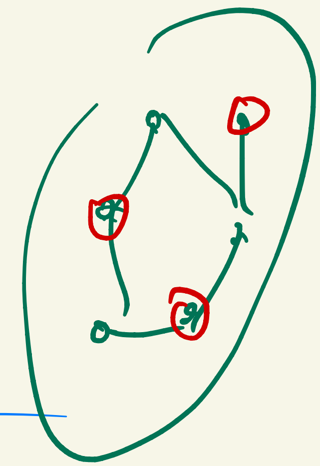
2023-11-14

DEF $G=(V,E)$ graph

$A \subseteq V$ is independent if

there are no edges within A,



i.e. $E \cap \binom{A}{2} = \emptyset$

i.e. $G[A]$ induced subgraph has no edges

i.e. $\overline{G}[A]$ is a clique

i.e. $(\forall u,v \in A)(u \not\sim v)$

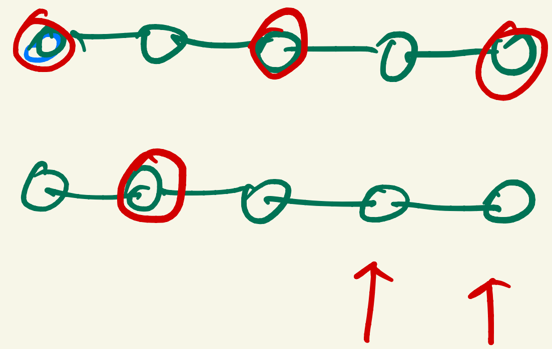
DEF INDEPENDENCE NUMBER of G

$$\alpha(G) = \max\{|A| \mid A \subseteq V,\ A \text{ is indep.}\}$$

\alpha

$$\alpha(P_n) = \left\lceil \frac{n}{2} \right\rceil$$

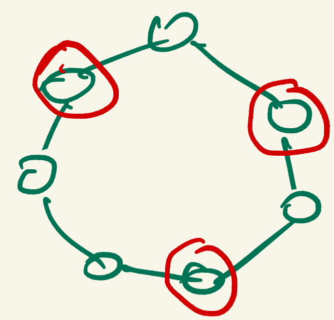
path of length $n-1$



P_5

$$\alpha(C_n) = \left\lfloor \frac{n}{2} \right\rfloor$$

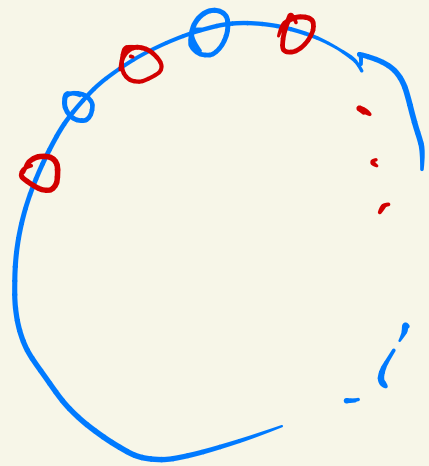
$n \geq 3$



C_7

NOTE: If A is an idep. set in G and $B \subseteq A$ then B is -"-

<u>Claim</u> $\alpha(C_n) = \left\lfloor \frac{n}{2} \right\rfloor$



Pf ① $\alpha(C_n) \geq \left\lfloor \frac{n}{2} \right\rfloor$

We just NEED to <u>find</u> an <u>indep</u> set of this size

DO greedy strategy finds it

To show that

$\alpha(G) \leq 100$

Want to show

$\alpha(G) \geq 100$

Pf: give a set A indep.

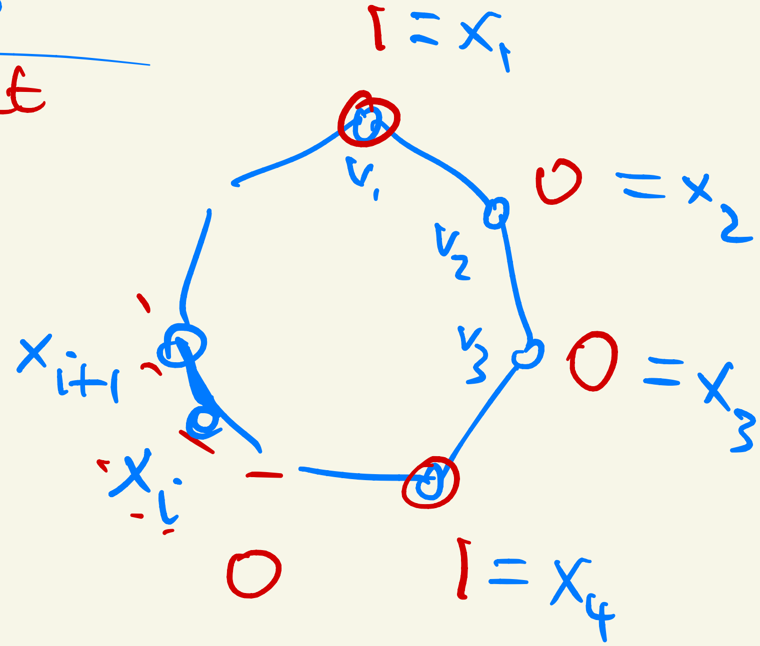
$|A| \geq 100$

② Claim $\alpha(C_n) \leq \lfloor \frac{n}{2} \rfloor$

Here we are up against every indep set

NTS: $(\forall A \subseteq V(C_n))(|A| \leq \frac{n}{2})$

Let A be an indep. set
(given to us by an adversary)



$$x_1 + x_2 \leq 1$$
$$x_2 + x_3 \leq 1$$
$$\vdots$$
$$x_{n-1} + x_n \leq 1$$
$$x_n + x_1 \leq 1$$

$$|A| = \sum_{i=1}^{n} x_i$$

. . .

$x_i \in \{0, 1\}$

$$2 \cdot \sum x_i \leq n$$

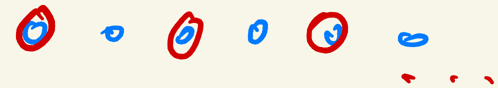
$$\therefore \quad \sum x_i \leq \frac{n}{2}$$

$$\therefore \quad |A| = \sum x_i \leq \lfloor \frac{n}{2} \rfloor$$

b/c $\sum x_i$ integer

LEMMA: $x \in \mathbb{Z}, r \in \mathbb{R}, x \leq r \implies x \leq \lfloor r \rfloor$

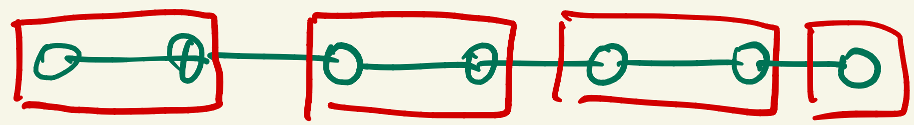
$$\alpha(P_n) = \left\lceil \frac{n}{2} \right\rceil$$



Pf $\quad \alpha(P_n) \geq \left\lceil \frac{n}{2} \right\rceil$

$\quad \alpha(P_n) \leq \left\lceil \frac{n}{2} \right\rceil$

Pf PHP



pigeon holes : $\left\lceil \frac{n}{2} \right\rceil$

injection $A \longrightarrow \{\text{pigeon holes}\}$ (at most 1 vertex of A in each P.H.)

$\therefore \quad |A| \leq$ # pigeon holes

EX $\alpha(G) \cdot \chi(G) \geq n$

EX If G is regular of deg ≥ 1

then $\alpha(G) \leq \lfloor \frac{n}{2} \rfloor$

$\boxed{\text{DO}}$ $\alpha(G) = n \iff G \cong \overline{K_n}$ no edges

$\alpha(G) = 1 \iff G \cong K_n$

$(\forall G)(1 \leq \alpha(G) \leq n)$

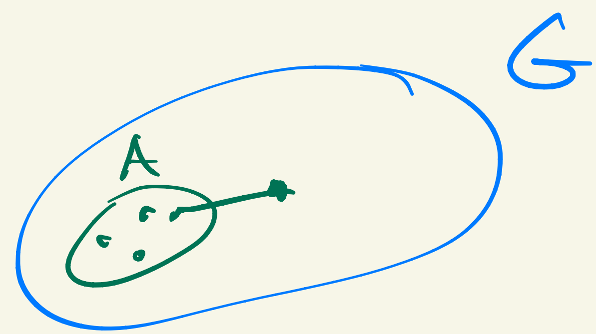
$\boxed{\text{EX}}$

$\forall n \geq 2$ Find connected graph G s.t. $\alpha(G) = n-1$

maximum indep. set: indep. set of max size
i.e., of size $\alpha(G)$

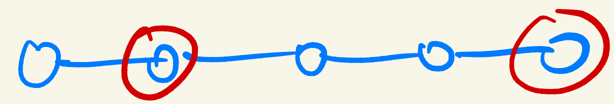
maximal —"— : indep set to which no vertex can be added

i.e. A is a maximal indep. set if
$(\forall B)(A \subseteq B \subseteq V \wedge B \text{ indep} \implies A = B)$



can be found efficiently by greedy strategy

Let $\beta(G)$ denote the size of the smallest maximal indep set



maximal but not maximum

EX $\beta(C_n)$

EX

$$\max\left\{ \frac{\alpha(G)}{\beta(G)} \;\middle|\; G \text{ connected graph of order } n \right\}$$

FINITE PROBABILITY SPACES

RIGOROUS DEF

Ω : a finite set

$\Omega \neq \emptyset$

DEF $P : \Omega \to \mathbb{R}$ is a probability distribution

if $(\forall a \in \Omega)(P(a) \geq 0)$

and

$$\sum_{a \in \Omega} P(a) = 1$$

(Ω, P) is a finite prob. space

Ω is a set, P is a function

INTUITION

experiment

Ω set of possible outcomes

example: dealing 5 cards "poker hand"

$|\Omega| = \binom{52}{5}$

flipping n coins HTTHTTT

$|\Omega| = 2^n$

shuffling a deck of n cards

$|\Omega| = n!$

DEF Event: subset of Ω

$$A \subseteq \Omega$$

EXTENDING domain of P to $\mathcal{P}(\Omega)$:

DEF $$P(A) := \sum_{a \in A} P(a)$$

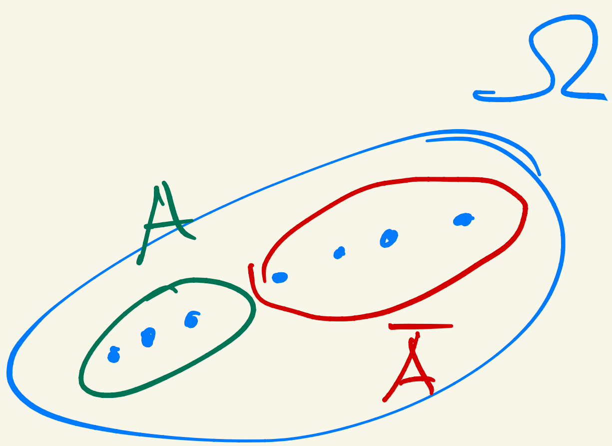
examples:

7 heads

full house

⋮

DEF elementary event: $\{a\}$ for $a \in \Omega$



$$(\forall A \subseteq \Omega)(0 \le P(A) \le 1)$$

$$A \subseteq B \Rightarrow P(A) \le P(B)$$

$$P(\bar{A}) = 1 - P(A)$$

$$\bar{A} = \Omega \setminus A$$

Example

$$\Omega = \{a, b, c\}$$

$$P(a) = \frac{1}{2}, \; P(b) = \frac{1}{3}, \; P(c) = \frac{1}{6}$$

probability distribution on Ω

event : $B = \{a, c\}$

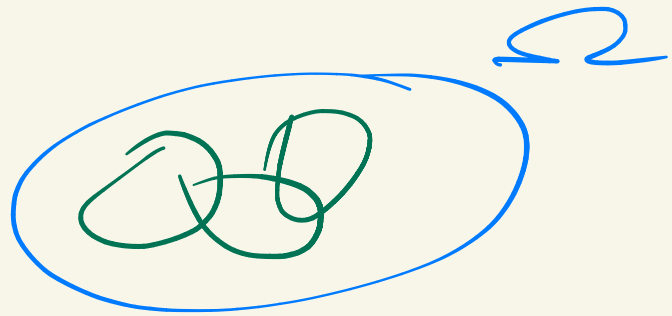
$$P(B) = P(a) + P(c) = \frac{1}{2} + \frac{1}{6} = \underline{\underline{\frac{2}{3}}}$$

MODULAR IDENTITY

$$P(A \cup B) + P(A \cap B) = P(A) + P(B)$$

$$P\left(\bigcup_{i=1}^{k} A_i\right) \leq \sum_{i=1}^{k} P(A_i)$$

UNION BOUND



DEF <u>Trivial</u> event:

$P(A) = 0$ or $P(A) = 1$

Examples: $P(\emptyset) = 0$ $\qquad P(\Omega) = 1$

<u>EX</u> # trivial events is a power of 2