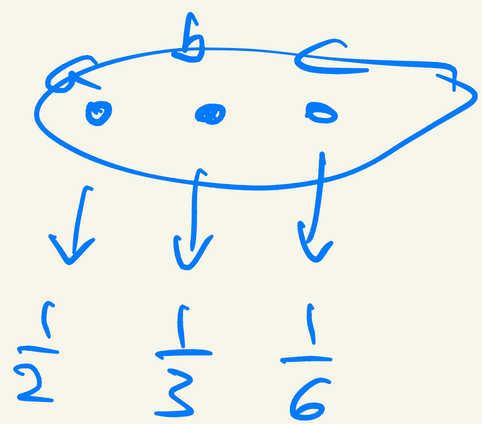
2023-11-16

Finite set Ω

Probability distribution on Ω:



function: $f : \Omega \to \mathbb{R}$

s.t. $(\forall a \in \Omega)(f(a) \geq 0)$

$$\sum_{a \in \Omega} f(a) = 1$$

we permit $f(a) = 0$

$\Omega \neq \emptyset$

Finite probability space:

(Ω, P)

finite set

prob. distribution on Ω

Ω "sample space" $a \in \Omega$ "elementary event"

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event: $A \subseteq \Omega$

powerset

extending P to larger domain: $\mathcal{P}(\Omega)$

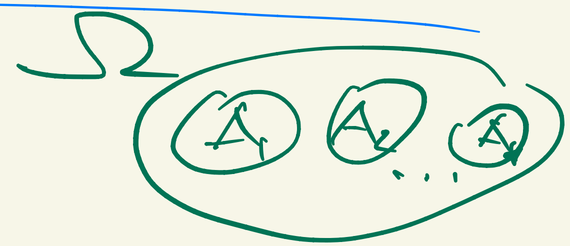
for $A \subseteq \Omega$ $P(A) := \sum_{a \in A} P(a)$ $P(\{a\}) = P(a)$

Uniform distribution:

$$(\forall a \in \Omega)\left(P(a) = \frac{1}{|\Omega|} \right)$$

Trivial event: $P(A) = 0$ or 1

Disjoint sets: $A \cap B = \emptyset$



Almost disjoint: $P(A \cap B) = 0$

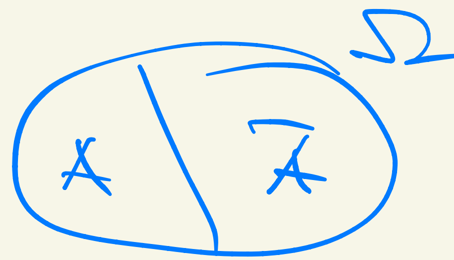
<u>Ex</u> If $A_1 \ldots A_k$ are pairwise almost disjoint then $P\left(\bigcup_{i=1}^{k} A_i\right) = \sum_{i=1}^{k} P(A_i)$

UNION BOUND

$\forall A_i \subseteq \Omega \qquad P\left(\bigcup_{i=1}^{k} A_i\right) \leq \sum_{i=1}^{k} P(A_i)$

$\overline{A} = \Omega \setminus A$

complement



$P(\overline{A}) = 1 - P(A)$

<u>DEF</u> A, B are <u>independent</u> if

$$P(A \cap B) = P(A) \cdot P(B)$$

A, B <u>positively</u> <u>correlated</u> if

$$P(A \cap B) > P(A) \cdot P(B)$$

negatively correlated $P(A \cap B) < P(A) \cdot P(B)$

DEF $A, B \subseteq \Omega, \quad P(B) > 0$

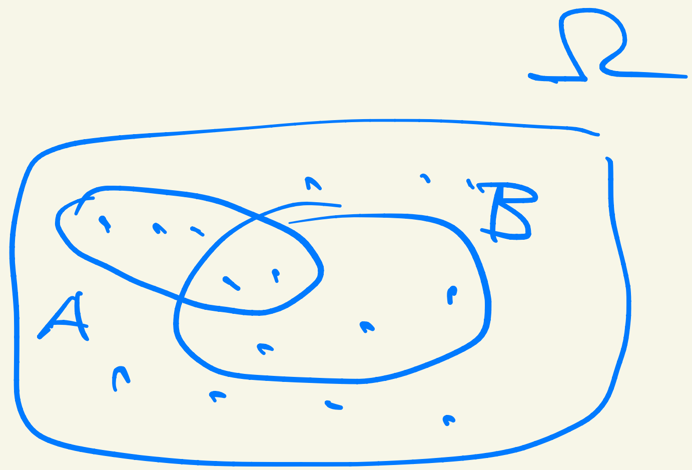
probability A given B:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

COROLLARY

If $a \in \Omega$

$$P(a|B) = \begin{cases} 0 & \text{if } a \notin B \\ \dfrac{P(a)}{P(B)} & \end{cases}$$



Pf: apply def of $P(A|B)$ to $A = \{a\}$

$A, B \subseteq \Omega$

when is $P(A) = P(A|B)$?

$$P(A) = \frac{P(A \cap B)}{P(B)}$$

$\Downarrow$ $\not\Uparrow$

$$P(A \cap B) = P(A) \cdot P(B)$$

T/F ? A, B are indep $\Longleftrightarrow P(A) = P(A|B)$

$\Longrightarrow$

$\not\Longleftarrow$ $P(B)$ can be 0

$\underline{\underline{\text{COR}}}$ If $P(B) \neq 0$ then A, B indep $\Longleftrightarrow P(A|B) = P(A)$

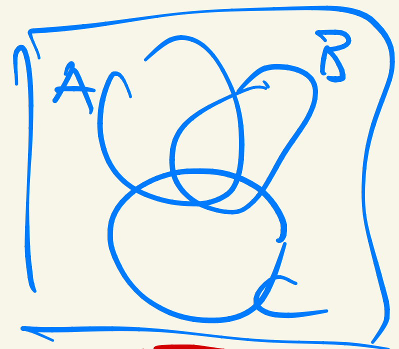
independence is a <u>symmetric relation</u> on $\mathcal{P}(\Omega)$

$\boxed{\text{DO}}$ If A is a trivial event then $(\forall B)(A, B$ are indep$)$

<u>DEF</u> $A, B, C \subseteq \Omega$ are (fully) independent if

① they are <u>pairwise</u> indep:

$$P(A \cap B) = P(A) \cdot P(B)$$
$$P(A \cap C) = P(A) \cdot P(C)$$
$$P(B \cap C) = P(B) \cdot P(C)$$



② $P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C)$

$\boxed{\text{DO}}$ ② $\not\Rightarrow$ ①

$\boxed{\text{HW}}$ Find a prob. space and 3 events that <u>pairwise</u> but not <u>fully</u> independent

① $\not\Rightarrow$ ②

Make $|\Omega|$ as small as possible

$\boxed{\text{HW}}$ If $\exists$ two nontrivial events then $|\Omega| \geq 4$

$\boxed{\text{HW}}$ If $\exists$ three nontriv events then $|\Omega| \geq 8$

If $\exists$ k indep events $\Longrightarrow |\Omega| \geq 2^k$

$\boxed{\text{HW}}$ If A, B indep then $A, \overline{B}$ also indep

If A, B, C " " $A, B, \overline{C}$ "

[it follows that $A, \overline{B}, \overline{C}$ ind, also $\overline{A}, \overline{B}, \overline{C}$ ind.]

$\boxed{\text{DO}}$ If $\left.\begin{array}{l} A, B \text{ indep} \\ C \text{ trivial} \end{array}\right\}$ then A, B, C indep

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STUDY DEF: $A_1 \ldots A_k$ indep

RANDOM VARIABLES

$$X : \Omega \longrightarrow \mathbb{R}$$

Set of random variables: $\mathbb{R}^{\Omega} = \{ f : \Omega \to \mathbb{R} \}$

we can take linear combinations of random variables: $3X + 5Y$

$$3X + 5Y - 10Z \qquad (X, Y, Z \text{ are r.v.'s})$$

Ω	X
a	$X(a)$
b	$X(b)$
c	$X(c)$
$\vdots$	$\vdots$

#1 Aggregate info

EXPECTED VALUE of X

$$E(X) = \sum_{a \in \Omega} X(a) \cdot P(a)$$

weighted average

value of fctn

weight

If P is uniform: $E(X) = \dfrac{\sum X(a)}{|\Omega|}$ = average

$\boxed{DO}$ $$E(cX) = c \cdot E(X)$$ scaling

$\boxed{DO}$ $$E(X+Y) = E(X) + E(Y)$$ additivity

(using distributivity)

$\therefore$ THEOREM (Linearity of expectation)

If $X_1 \cdots X_k$ are rv's and $c_i \in \mathbb{R}$

then $$E\left(\sum_{i=1}^{k} c_i X_i\right) = \sum_{i=1}^{k} c_i E(X_i)$$

Flipping k coins

$$X_i = \begin{cases} 1 & \text{if } i^{th} \text{: HEADS} \\ 0 & \text{if } i^{th} \text{: TAIL} \end{cases}$$

$$Z := \#\text{ heads} = \sum X_i$$

$$E(Z) = \sum E(X_i) = \frac{k}{2}$$

$$E(X_i) = \frac{1}{2}$$

HHTHTTT

1 1 0 1 0 0 0

DEF an **indicator variable**:

takes values 0 or 1 only

thm If X is an indicator variable

then $E(X) = P(X=1)$

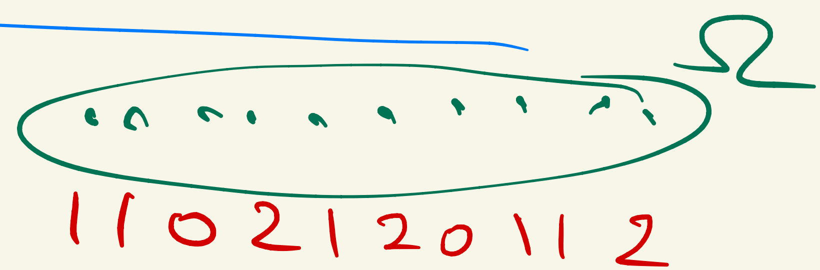
DO

THM

$$E(X) = \sum_{y \in \mathbb{R}} y \cdot P(X=y) = \sum_{y \in \mathrm{range}(X)} y \cdot P(X=y)$$

$$\text{“}X=y\text{”} = \{a \in \Omega \mid X(a) = y\}$$

Pf $E(X) = \sum_{a \in \Omega} X(a) P(a)$



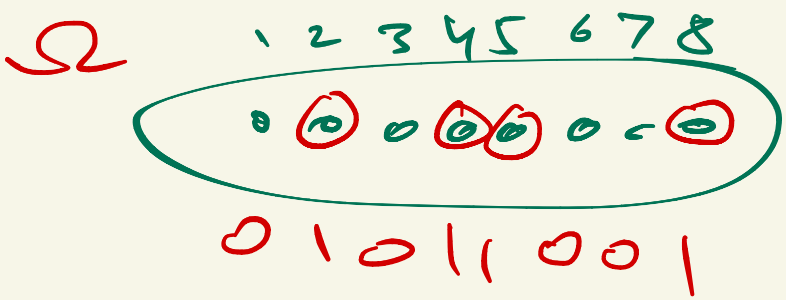
In particular, if $\mathrm{range}(X) \subseteq \{0,1\}$

then

$$E(X) = 0 \cdot P(X=0) + 1 \cdot P(X=1) = P(X=1)$$

One-to-one corresp.

events $\longleftrightarrow$ indicator variables



$A = \{2, 4, 5, 8\}$

$$\vartheta_A(a) = \begin{cases} 1 & \text{if } a \in A \\ 0 & \text{if } a \notin A \end{cases}$$

\vartheta

$$E(\vartheta_A) = P(A)$$

!

biased coins: $P(\text{heads}) = p$

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k coin flips $\Longrightarrow E(\#\text{heads}) = k \cdot p$

Pf Same as before:

$$X_i = \begin{cases} 1 & \text{if } i\text{th coin Heads} \\ 0 & \text{Tails} \end{cases}$$

$$\#\text{heads} = Z = \sum X_i$$

$$E(Z) = \sum E(X_i) = \sum_{i=1}^{k} p = k \cdot p$$