# PROBLEM SESSION 2023-11-17

14.65 6-color theorem: $\forall$ planar graph is 6-colorable

(1) If $G$ is planar and $H \subseteq G$ then $H$ is planar

(2) If $G$ is planar, $n \geq 1$, then $\exists$ vertex of deg $\leq 5$

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Pf induction on $n$

$n = 1$ ✓

$n \geq 2$, assume true for $n-1$ vertices ← IH

$G$ has $n$ vertices, pick a vertex $x$ of deg $\leq 5$

$G - x$ 6-colorable by IH

<u>Claim</u> Any 6-coloring of $G-x$ can be <u>extended</u> to a 6-coloring of $G$.

Pf Pick a 6-coloring of $G-x$. Now color $x$ by a color not used by its neighbors. ✓

14.89

$$\binom{n}{5} = \frac{\overbrace{n(n-1)\cdots(n-4)}^{}}{5!} = \frac{N}{5!}$$

DON'T

$$\binom{n}{5} = \frac{n!}{5!\,(n-5)!}$$

true but ugly

$$\frac{N}{n^5} = \frac{n(n-1)\cdots(n-4)}{n^5} = \underset{\underset{1}{\downarrow}}{\left(1-\frac{1}{n}\right)}\underset{\underset{1}{\downarrow}}{\left(1-\frac{2}{n}\right)}\underset{\underset{1}{\downarrow}}{\left(1-\frac{3}{n}\right)}\underset{\underset{1}{\downarrow}}{\left(1-\frac{4}{n}\right)} \longrightarrow 1$$

alternative ~~proof~~:

$N$ is a poly of deg 5 in $n$ with leading term $n^5$

$\therefore\ N \sim n^5$ by a general result

$$\therefore\ N \sim n^5$$

$$\therefore\ \binom{n}{5} = \frac{N}{5!} \sim \frac{n^5}{5!} = \frac{n^5}{120}$$

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$$\left(1-\frac{1}{n}\right)^n = \underset{\underset{1}{\downarrow}}{\left(1-\frac{1}{n}\right)}\underset{\underset{1}{\downarrow}}{\left(1-\frac{1}{n}\right)}\cdots\underset{\underset{1}{\downarrow}}{\left(1-\frac{1}{n}\right)} \longrightarrow e$$

**14.96** $f(x) = a x^k + bx^{k-1} + \dots \quad ab \neq 0$

then $\underline{f(n) \sim a n^k}$

**Pf** $$\frac{f(n)}{n^k} = a + \underset{\substack{\downarrow \\ 0}}{\frac{b}{n}} + \underset{\substack{\downarrow \\ 0}}{\frac{\cdot\cdot}{n^2}} + \underset{\substack{\downarrow \\ 0}}{\dots} \longrightarrow a$$

$$\frac{f(n)}{a \cdot n^k} \longrightarrow 1 \qquad f(n) \sim a n^k$$

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**14.97** $$\frac{f(x)}{g(x)} = \frac{a x^k + \dots}{c x^l + \dots}$$

$$\frac{f(n)}{g(n)} \sim \frac{a n^k}{c n^l} = \underline{\underline{\frac{a}{c} \cdot n^{k-l}}}$$

14.101

$$\sqrt{n^2+1}-n = \frac{1}{\underbrace{\sqrt{n^2+1}+n}} \sim \frac{1}{2n}$$

$$\sqrt{n^2+1} \sim n$$

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$$\sqrt{n^2+1}+n \sim 2n$$

Divide by $\underline{\underline{n}}$: $\sqrt{1+\underset{\underset{0}{\downarrow}}{\frac{1}{n^2}}}+1 \longrightarrow 2$

$$\therefore \quad \frac{\sqrt{n^2+1}+n}{2n} \longrightarrow 1$$

✓

14.99

$$\ln\left(1+\frac{1}{n}\right) \sim \frac{1}{n}$$

~~Apply L'Hôpital to $\frac{\ln(1+\frac{1}{n})}{1/n}$~~

**Lemma**

$$\lim_{x\to 0}\frac{\ln(1+x)}{x} = \lim_{x\to 0}\frac{\frac{1}{1+x}}{1} = \lim_{x\to 0}\frac{1}{1+x} = 1$$

then follows by setting $x=\frac{1}{n}$

Alternative ~~proof~~ of Lemma: instead of L'Hôpital, use def of derivative:

$$f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

Let $f(x)=\ln x$ $\quad f'(1) = \lim_{h\to 0}\frac{\ln(1+h)-\overbrace{\ln(1)}^{0}}{h}$

$$\therefore \lim_{h\to 0}\frac{\ln(1+h)}{h} = \ln'(1) = \frac{1}{t}\Big|_{t=1} = 1 \checkmark$$

14.99 $\sin\left(\frac{1}{n}\right) \sim \frac{1}{n}$

LEMMA

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

b/c this limit is $(\sin)'(0) = \cos 0 = 1$ ✓

(also works by L'Hôpital)

14.105 $a_n \sim b_n \not\Rightarrow a_n^n \sim b_n^n$

Counterexample: $a_n = 1$, $b_n = 1 + \frac{1}{n}$

$a_n^n = 1$, $b_n^n = \left(1 + \frac{1}{n}\right)^n \to e$

14.131

$$a_n, b_n > 1$$

$$a_n \sim b_n \quad \not\Longrightarrow \quad \ln a_n \sim \ln b_n$$

Pf : need counterexample : pair of sequences that satisfies $a_n \sim b_n$ but $\ln a_n \not\sim \ln b_n$

$$\left.\begin{array}{l} a_n := e^{1/n} \longrightarrow 1 \\ b_n := e^{2/n} \longrightarrow 1 \end{array}\right\} \Longrightarrow a_n \sim b_n$$

$$\begin{array}{l} \ln a_n = \frac{1}{n} \\ \ln b_n = \frac{2}{n} \end{array} \qquad \text{quotient} = \frac{1}{2} \longrightarrow \frac{1}{2} \qquad ✓$$

another example: $a_n = 1 + \frac{1}{n} \longrightarrow 1$

$b_n = 1 + \frac{2}{n} \longrightarrow 1 \quad \therefore \; a_n \sim b_n$

$$\left.\begin{array}{l} \ln a_n \sim \frac{1}{n} \\ \ln b_n \sim \frac{2}{n} \end{array}\right\} \Longrightarrow \frac{\ln a_n}{\ln b_n} \sim \frac{1}{2}, \text{ i.e. } \frac{\ln a_n}{\ln b_n} \longrightarrow \frac{1}{2}$$

14.37 Greedy coloring uses $\le 1+\Delta$ colors

$\Delta = \deg_{max}$

Pf induction on $n$. True if $n=1$. Now $n \ge 2$

IH true for $n-1$.

G has $n$ vertices

↗

input, given to us by an adversary

pick any $x \in V(G)$

$\Delta(G-x) \le \Delta(G)$

so by IH $G-x$ is colorable by $\Delta(G-x)$ colors $\therefore$ by $\Delta(G)$ colors

Pick a $\Delta$-coloring of $G-x$

Claim This coloring extends to G

Pf: color $x$ by a color not used by any neighbor. ✓

n even

14.39 Find <u>bipartite graph</u> s.t. greedy col. uses $n/2$ colors

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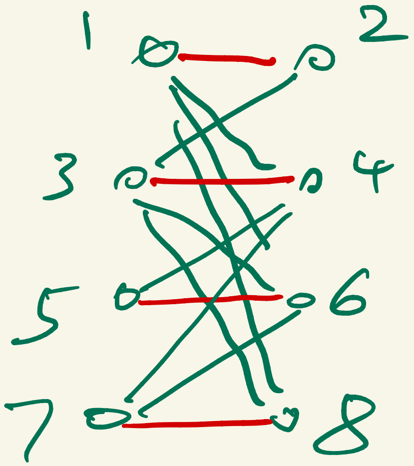
<u>Pf</u> $V = [n]$

for $k = 1, \ldots, \frac{n}{2}$

$2k-1$ is adjant to $2j$ for all $j \neq k$



<u>Claim</u> If $c$ denotes greedy coloring then

$(\forall k)(c(2k-1) = c(2k) = k)$

<u>Pf</u> by induction $k$

Base: $k=1$ ✓

IH: true for all $k' < k$

Pf for $k$: vx $2k-1$ is adj. to $2j$ for $j \leq k-1$ – these have all colors $1, \ldots, k-1$ so $c(2k-1) = k$ ← forced

Same for vx $2k$ ✓

$r \geq 2s$ Kneser's graph

$$V = V(Kn(r,s)) = \binom{[r]}{s}$$

$A, B \in V$ $A \sim B$ if $A \cap B = \emptyset$

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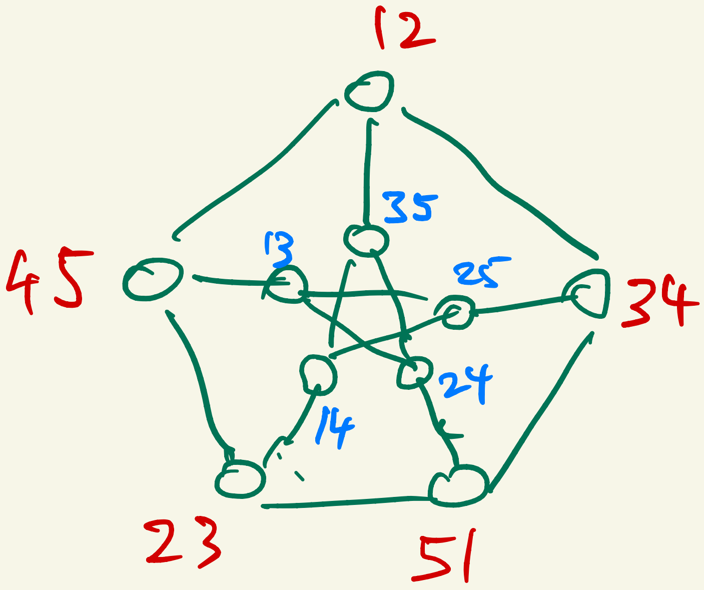
$Kn(5,2) \cong$ Petersen's



14.49

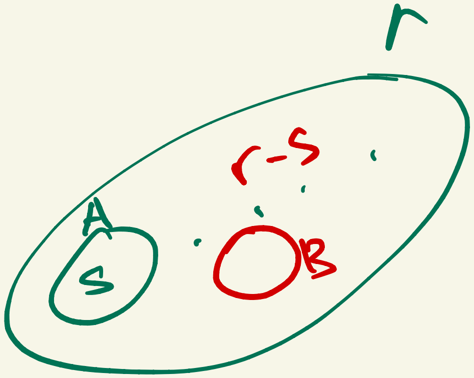
$Kn(r,s)$ is regular of deg $\binom{r-s}{s}$

$|A| = s$

$A \subseteq [r]$ count $B \subseteq [r]$

$|B| = s$

$A \cap B = \emptyset$



14.53 $\chi(Kn(r,s)) \le r-2s+2$

vertices of each color must be an independent set

indep. set in $K_n$ corresponds to s-subsets that pairwise intersect

$C_1$ : all s-subsets containing "1"

$C_2$ : – " – not containing "1" but containing "2"

⋮

$C_k$ : – " – not containing $1, \dots, k-1$ – " – "k"

for $k = 1, \dots, r-2s+1$

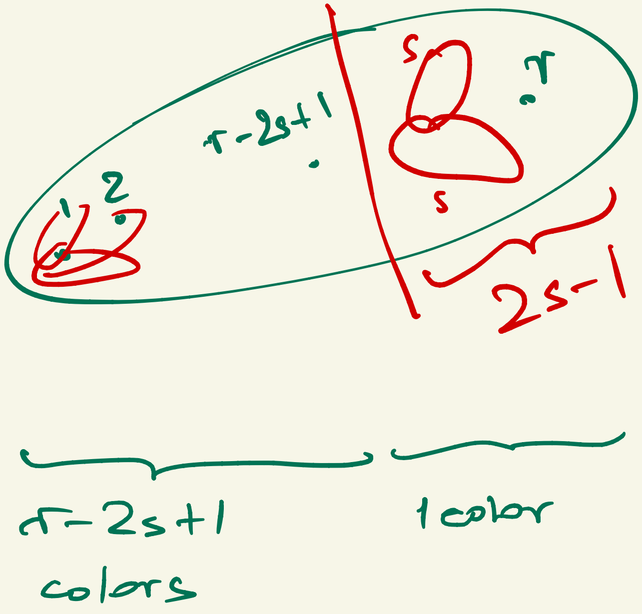
remaining:

s-subsets of $\{\underbrace{r-2s+2, \dots, r}_{2s-1 \text{ elements}}\}$

↓

pairwise intersect

⟶ get 1 color



If $a_n \sim b_n$ then ~~$\lim a_n = \lim b_n$~~

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15.47 $(\ln n)^{100} = o(n)$

NTS $\frac{(\ln n)^{100}}{n} \longrightarrow 0$

i.e. $\frac{\ln n}{n^{1/100}} \to 0$

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15.45 **DO** $(\forall c>0)\ \lim_{x\to\infty} \frac{\ln x}{x^c} = 0$

$\Downarrow$

15.47 : $c := \frac{1}{100}$

bootstrapping

Pf: L'Hôpital **once**

$$\frac{1/x}{c\,x^{c-1}} = \frac{1}{c}\cdot\frac{1}{x^c} \longrightarrow 0$$

✓

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Actually 15.43 $\Longrightarrow$ 15.45 $\forall c>0$

$\frac{\ln x}{x} \to 0$ $\quad x\to\infty$ $\Longrightarrow$ $\frac{\ln y}{y^c} \longrightarrow 0$

$\frac{\ln y}{y^c} = \frac{1}{c}\cdot\frac{\ln x}{x}$

use 15.43 with $x = y^c$

$\ln x = c\cdot\ln y$

$\ln y = \frac{1}{c}\ln x$

"exponential growth beats poly. growth"

$f(x) = e^x$

$g(x) = x^C$

then $\frac{g(n)}{f(n)} \to 0$

Example: $e^{\sqrt{x}}$

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Pf case $\underline{c=1}$ $\otimes$

$\frac{x^C}{e^x} \overset{?}{\longrightarrow} 0$

$y := e^x$

$x = \ln y$

$\frac{(\ln y)^C}{y} \to 0$ ✓

Case $\underline{\text{general}}$ $c>0$

$\frac{x^C}{e^{x^c}} \overset{?}{\longrightarrow} 0$

$z := x^c$

$x = z^{1/c}$

$\frac{z^{C/c}}{e^z} \longrightarrow 0$ b/c $\otimes$

15.15 Given $(b_n)$ $\exists (a_n)$ s.t. $a_n = o(b_n)$

$\iff$ $b_n$ is eventually nonzero

Pf. $\Longrightarrow$ ASSN $a_n = o(b_n)$, i.e. $\frac{a_n}{b_n} \to 0$

NTS $b_n$ event. nonz.

Pf by contradiction: $b_n = 0$ inf. often $\therefore$ $\frac{a_n}{b_n}$ is undefined inf. often $\therefore$ $\nexists \lim \frac{a_n}{b_n}$

$\Longleftarrow$ ASSN $b_n$ event. nonzero:

$(\exists n_0)(\forall n > n_0)(b_n \neq 0)$

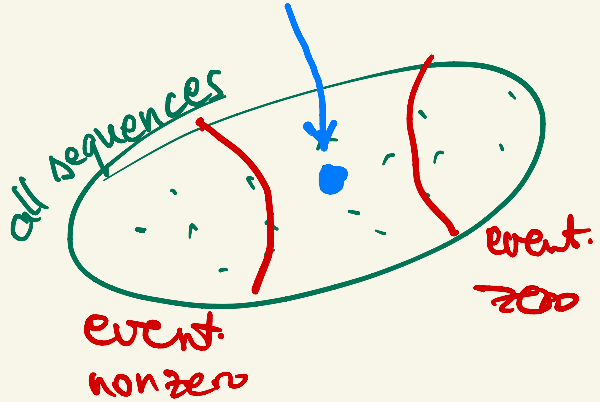
NEED $(a_n)$ s.t. $\frac{a_n}{b_n} \to 0$ Let $a_n = 0$

$\to$ $b_n = 0$ infinitely often

we find sequence that is neither 010101

$\neg$ ($b_n$ is eventually nonzero)

is NOT "$b_n$ is eventually zero"



ASSNs

14.129 $a_n, b_n > 1$

$a_n \sim b_n$

$a_n$ bded away from 1

DC $b_n$ bded away from 1

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ASSN $\underline{(\exists\, \varepsilon>0)}(\exists n_0)(\forall n)(n>n_0 \Longrightarrow a_n > 1+\varepsilon)$

NTS $(\exists\, \delta>0)(\exists n_1)(\forall n)(n>n_1 \Longrightarrow \underline{b_n > 1+\delta})$

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Claim ~~$n_1 := n_0$~~, any $0<\delta<\varepsilon$ works

Pf **Want** $\underline{b_n > \frac{1+\delta}{1+\varepsilon} a_n}$ this will suffice if $\underline{n > n_0}$

$\Downarrow$

$\longrightarrow b_n > 1+\delta \Longleftarrow a_n > 1+\varepsilon$

$\frac{1+\delta}{1+\varepsilon} =: 1-\gamma \quad \gamma>0$

$a_n \sim b_n \Longrightarrow (\exists n_2)(\forall n)(n>n_2 \Longrightarrow \frac{b_n}{a_n} > 1-\gamma)$ $n_1 := \max(n_0, n_2)$