2023-11-20

1

REVIEW:

Finite Probability Space

$$(\Omega, P)$$

Ω ↑ finite set

P ↑ function

$P : \Omega \to \mathbb{R}$

domain codomain

such that

P is a probability distibution

i.e. $P : \Omega \to \mathbb{R}$

s.t. ① $(\forall a \in \Omega)(P(a) \geq 0)$

② $\sum_{a \in \Omega} P(a) = 1$

$\therefore \ P(a) \leq 1$

2

$$\Omega = \{a, b, c, d\}$$

Ω	P	X	Y
a	0.1	1	-5
b	0.2	2	0
c	0.6	6	42
d	0.1	5000	7

RANDOM VARIABLE:

FUNCTION

$$X : \Omega \to \mathbb{R}$$

no constraint!

$$\min X \le E(X) \le \max X$$

$\min Y = -5$

$\max Y = 42$

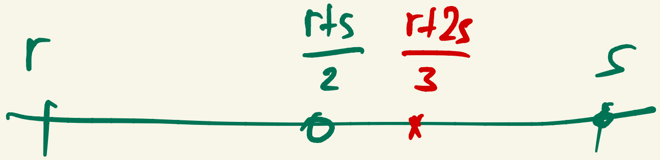
$$E(X) = \sum_{a \in \Omega} X(a) \cdot P(a)$$

$E(X) \approx 500$

$$E(Y) = Y(a) \cdot P(a) + Y(b) P(b) + Y(c) P(c) + Y(d) P(d)$$

$$= (-5) \cdot 0.1 + 0 \cdot 0.2 + 42 \cdot 0.6 + 7 \cdot 0.1$$

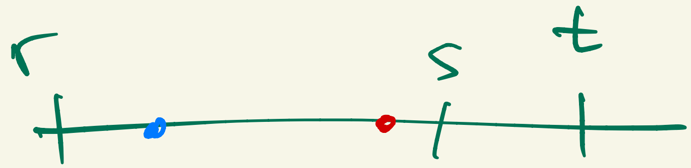
$$= -0.5 + 25.2 + 0.7 = \underline{\underline{25.4}}$$



$\frac{1}{2}, \frac{1}{2}$ $\qquad$ $\frac{1}{2}r + \frac{1}{2}s$

$\frac{1}{3}, \frac{2}{3}$ $\qquad$ $\frac{r}{3} + \frac{2s}{3}$

$0, 1$ $\qquad$ s



$\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$ $\qquad$ $\frac{r+s+t}{3}$

$0.8, 0.1, 0.1$

Ω	P	X	Y
a	0.1	1	-5
b	0.2	2	0
c	0.6	6	42
d	0.1	5000	7

"$X>2$" event $= \{c, d\}$

$$P(X>2) = P(c) + P(d) = 0.7$$

$$P(Y \text{ is odd}) = P(a) + P(d) = 0.2$$

Q: "$X>2$" and "Y is odd" $\left\{\begin{array}{l}\text{indep} \\ \text{pos} \\ \text{neg}\end{array}\right\}$ correl ?

$$Y(a) = -5$$
$$Y(b) = 0$$
$$Y(c) = 42$$
$$Y(d) = 7$$

$$P(X>2 \wedge Y \text{ is odd}) = P(d) = \underline{0.1}$$

compare with

$$P(X>2)\cdot P(Y \text{ is odd}) = 0.7 \cdot 0.2 = \underline{0.14}$$

$\therefore$ negatively correlated

DEF RVs X, Y are

positively correlated if $E(XY) - E(X)E(Y) > 0$

uncorrelated if $E(XY) - E(X)E(Y) = 0$

negatively correlated if $E(XY) - E(X)E(Y) < 0$

Ω	P	X	Y	XY
a	0.1	1	−5	−5
b	0.2	2	0	0
c	0.6	6	42	252
d	0.1	5000	7	35,000

negatively correlated

$$E(XY) = (-5)\cdot 0.1 + 0 + 252\cdot 0.6 + 35{,}000 \cdot 0.1 = \underline{3650.7}$$

$$E(X)\cdot E(Y) = (0.1 + 0.4 + 3.6 + 500)(-0.5 + 25.2 + 0.7) = \underline{504.1}\cdot\underline{25.4} = \underline{12{,}804.14}$$

DEF X, Y are indep. if
$(\forall x, y \in \mathbb{R})$(events "$X = x$" and "$Y = y$" are independent)

THM Indep $\Longrightarrow$ uncorrelated
$\not\Longleftarrow$

FIND counterexample to $\Longleftarrow$

NEED TO FIND (Ω, P) and RV's X, Y s.t.
X, Y are UNCORRELATED but not INDEP

$\Omega = \{a, b, c\}$. P uniform

Ω	X	Y
a	x_1	y_1
b	x_2	y_2
c	x_3	y_3

$$E(X) = \frac{x_1 + x_2 + x_3}{3} \qquad E(Y) = \frac{y_1 + y_2 + y_3}{3}$$

UNCORR

$$E(XY) = \frac{x_1 y_1 + x_2 y_2 + x_3 y_3}{3} = \frac{x_1 + x_2 + x_3}{3} \cdot \frac{y_1 + y_2 + y_3}{3}$$

NEED TO FIND (Ω, P) and RV's X, Y s.t.
X, Y are UNCORRELATED but <u>not</u> INDEP

$\Omega = \{a, b, c\}$. P uniform

Ω	X	Y
a	x_1	y_1
b	x_2	y_2
c	x_3	y_3

$$E(X) = \frac{x_1 + x_2 + x_3}{3} \qquad E(Y) = \frac{y_1 + y_2 + y_3}{3}$$

$$E(XY) = \frac{x_1y_1 + x_2y_2 + x_3y_3}{3} \overset{\text{UNCORR}}{=} \frac{x_1 + x_2 + x_3}{3} \cdot \frac{y_1 + y_2 + y_3}{3}$$

$$3(x_1y_1 + x_2y_2 + x_3y_3) = (x_1 + x_2 + x_3)(y_1 + y_2 + y_3)$$

A SOLUTION: $x_1 = \dots = y_3 = 1$ ~ indep ✗

$$\boxed{x_1 = x_2 = 0 \quad x_3 = 1}$$

$$3y_3 = y_1 + y_2 + y_3$$

$$2y_3 = y_1 + y_2$$

"$X=0$" $= \{a, b\}$

A SOL: $\boxed{\begin{matrix} y_1 = 1 \\ y_2 = 5 \\ y_3 = 3 \end{matrix}}$

$$\underbrace{P(X=0 \wedge Y=3)}_{0} \text{ vs. } \underbrace{P(X=0)}_{\frac{2}{3}} \cdot \underbrace{P(Y=3)}_{\frac{1}{3}}$$

$\{a, b\} \cap \{c\} = \emptyset$

$0 \neq \frac{2}{3} \cdot \frac{1}{3}$

NOT INDEP.

$|A| = n$

$B \subseteq A \qquad |B| = r$

pick random B : sample space size $\binom{n}{r}$

$C \subseteq A \qquad |C| = s$

pick random B, C : sample space size $\binom{n}{r} \cdot \binom{n}{s}$

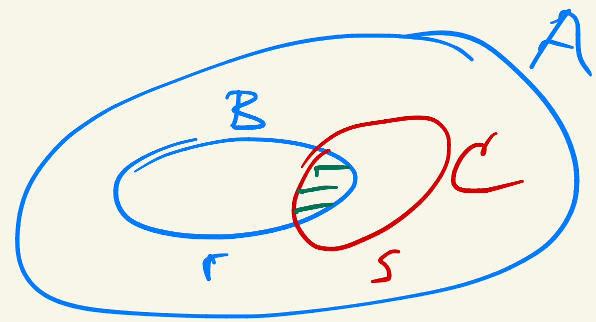
uniform

$E(|B \cap C|)$

↑ RV

"$i \in B \cap C$" = "$i \in B$" $\cap$ "$i \in C$"

P: $r/n \cdot s/n$



$$|B \cap C| = \sum_{i \in A} X_i$$

⊛ $X_i = \begin{cases} 1 & \text{if } i \in B \cap C \\ 0 & \text{o/w} \end{cases}$ ← $i \in A \setminus (B \cap C)$

$$E(|B \cap C|) = \sum_{i \in A} E(X_i) = \sum_{i \in A} P(X_i = 1) = \sum_{i \in A} P(i \in B \cap C) = \sum_{i \in A} \frac{rs}{n^2} = \boxed{\frac{rs}{n}} \checkmark$$

$$P(i \in B) = \frac{\binom{n-1}{r-1}}{\binom{n}{r}} = \frac{r}{n} \leftarrow \boxed{\text{DO}}$$