2023-11-28

1

(Ω, P)

Ω finite set

P function $\Omega \to \mathbb{R}$

probability distribution

$(\forall a \in \Omega)(P(a) \geq 0)$

$$\sum_{a \in \Omega} P(a) = 1$$

event $A \subseteq \Omega$

$$P(A) := \sum_{a \in A} P(a)$$

Random variable

$X : \Omega \to \mathbb{R}$

no constraint

2

DEF X, Y are independent if

$$(\forall x, y \in \mathbb{R})\left(P(X = x \wedge Y = y) = P(X=x) \cdot P(Y=y) \right)$$

i.e. $(\forall x, y \in \mathbb{R})$ the events "$X=x$" and "$Y=y$" are independent

THM If X, Y are indep. then

DO

$$E(XY) = E(X)E(Y)$$

DEF X, Y are

<u>positively correlated</u> if $E(XY) > E(X)\cdot E(Y)$ i.e. $Cov(X,Y)>0$

<u>uncorrelated</u> if $=$ $Cov(X,Y)=0$

<u>negatively correlated</u> if $<$ $Cov(X,Y)<0$

THM restated:

X, Y indep $\Longrightarrow$ uncorrelated

$\Longleftarrow$ if $|\Omega| = 2$

$\boxed{\text{DO}}$ $\not\Longleftarrow$ DO: with $|\Omega|=3$, uniform, X, Y range $\subseteq \{-1,0,1\}$

Covariance of X,Y $\quad Cov(X,Y) = E(XY) - E(X)E(Y)$

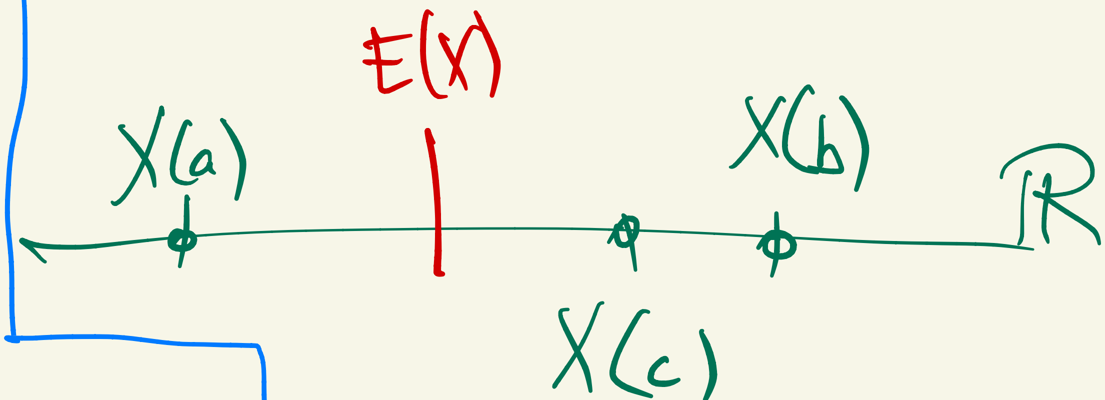
2nd aggregate value for RV

DEF Variance of X

$$\mathrm{Var}(X) = E\left[(X - E(X))^2\right]$$

EXAMPLE: $\Omega = \{a, b, c\}$

Ω	X	P
a	x_1	p_1
b	x_2	p_2
c	x_3	p_3

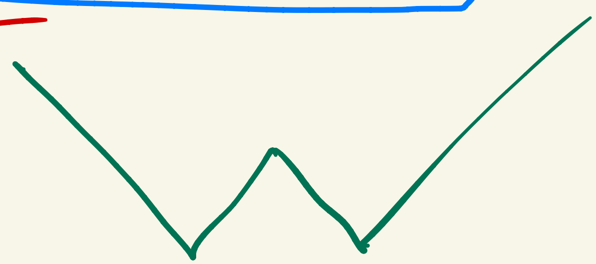


$$m = E(X) = p_1x_1 + p_2x_2 + p_3x_3$$

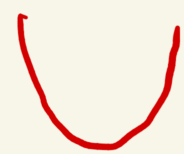
$$\mathrm{Var}(X) = p_1(x_1-m)^2 + p_2(x_2-m)^2 + p_3(x_3-m)^2$$

$$= \underbrace{p_1x_1^2 + p_2x_2^3 + p_3x_3^2}_{E(X^2)} - m^2$$

← THM from next page



$|x-a| + |x-b|$



$(x-a)^2 + (x-b)^2 + \cdots + (x-k)^2$

$\circledast$ $\mathrm{Var}(X) = E\left[(X - E(X))^2\right] = E\left[(X-m)^2\right]$

$m := E(X)$

$$= E\left(X^2 - 2mX + m^2\right)$$

$$= E(X^2) - 2m\underbrace{E(X)}_{m} + m^2 = E(X^2) - m^2$$

$$\therefore \quad \boxed{\mathrm{Var}(X) = E(X^2) - (E(X))^2}$$

By $\circledast$ def $\quad \mathrm{Var}(X) \geq 0$

$$\therefore \quad \boxed{E(X^2) \geq E(X)^2}$$

CAUCHY-SCHWARZ INEQUALITY

MARKOV'S INEQUALITY

If $X \geq 0$ (meaning $(\forall a \in \Omega)(X(a) \geq 0)$)

non-negative R.V.

$(\forall s \geq 0)(E(X) \geq s \cdot P(X \geq s))$

$\forall s > 0$ $P(X \geq s) \leq \frac{E(X)}{s}$

then

$$(\forall t > 0)\left(P\left(X \geq \underbrace{t \cdot E(X)}_{s}\right) \leq \frac{1}{t}\right)$$

Proof

$$E(X) = \sum_{a \in \Omega} X(a) \cdot P(a) \geq$$

$$\sum_{\substack{a \in \Omega \\ X(a) \geq s}} X(a) \cdot P(a) \geq \sum_{\substack{a \in \Omega \\ X(a) \geq s}} s \cdot P(a) = s \cdot P(X \geq s)$$

CHEBYSHEV'S INEQUALITY

X arbitrary RV — no constraint

$m := E(X)$

a CONCENTRATION INEQUALITY

$$(\forall r>0)\quad \boxed{P(|X-m| \geq r) \leq \frac{Var(X)}{r^2}} = \left(\frac{\sigma(X)}{r}\right)^2$$

Proof. $P(|X-m| \geq r) = P(\underbrace{(X-m)^2}_{Z \geq 0} \geq r^2) \leq \frac{E(Z)}{r^2}$ $\quad E(Z) = Var(X)$

Apply MARKOV to Z

Standard deviation $\sigma(X) = \sqrt{Var(X)}$

\sigma

Variance of <u>indicator</u> variable

$$P(X=1) = p$$
$$P(X=0) = 1-p$$

$$\to X^2 = X$$

$$E(X) = p \qquad E(X^2) = E(X) = p$$

$$Var(X) = E(X^2) - E(X)^2 = p - p^2 = p(1-p)$$

$$\boxed{Var\, X = p(1-p)}$$

$$Var(X) = Cov(X,X)$$

$Cov(X,Y) =$
$E(XY) - E(X)E(Y)$

Let $Y = X_1 + \cdots + X_k$

then $E(Y) = \sum_i E(X_i)$

$$\boxed{\begin{aligned} Var(Y) &= \sum_i \sum_j Cov(X_i, X_j) \\ &= \sum_i Var(X_i) + 2 \sum_{i<j} Cov(X_i, X_j) \end{aligned}}$$

COR If the X_i are pairwise uncorrelated then

$$Var(Y) = \sum_i Var(X_i)$$

$Y_1 \ldots Y_k$: <u>indicator</u> <u>variables</u>

$P(Y_i = 1) = p$

(biased coin flips
$P(\text{heads}) = p$)

$X = \sum_i Y_i$

<u># heads
in k coin flips</u>

$E(X) = k \cdot p$

$Var(X) = k \cdot p(1-p)$

assuming the Y_i
are <u>pairwise indep</u>

$\boxed{\text{DO}}$ A, B events, ϑ_A, ϑ_B corresponding indicator variables

then A, B are pos. correl / indep / neg. correl $\iff \vartheta_A, \vartheta_B$ < / : / >

DEF X, Y, Z are <u>indep</u> if

$$(\forall x, y, z)\big(P(X=x \wedge Y=y \wedge Z=z) = P(X=x)\cdot P(Y=y)\cdot P(Z=z)\big)$$

i.e. the events "$X=x$", "$Y=y$", "$Z=z$" are indep

DO If X, Y, Z indep then they are pairwise indep.

DEF $X_1 \dots X_k$ are indep. if

$$(\forall x_1 \dots x_k \in \mathbb{R})\left(P(X_1=x_1 \wedge \dots \wedge X_k=x_k) = \prod_{i=1}^{k} P(X_i=x_i)\right)$$

THM If $X_1 \ldots X_k$ are indep

then $E\left(\prod X_i\right) = \prod E(X_i)$

$\boxed{\text{HW}}$ due Fri

poker hand $\quad X = \#$ Aces

$\quad Y = \#$ Spades

$\boxed{\text{HW}}$ X, Y not independent

$\boxed{\text{XC}}$ X, Y are uncorrelated: do it for a deck of $k \cdot \ell$ cards $(k=4, \ell=13)$ hand: r cards

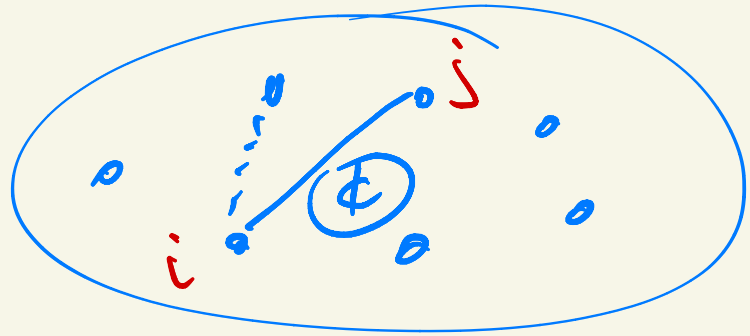
RANDOM GRAPHS:

The ERDŐS–RÉNYI model

1960

$G(n,p)$ model $\quad 0<p<1$

graph defined by $\binom{n}{2}$ independent flips of a p-biased coin: $\quad P(i \sim j) = p$



$\boxed{\text{DO}}$ $E(\#\text{edges}) = \binom{n}{2} p$

$\boxed{\text{DO}}$ $\mathrm{Var}(\#\text{edges}) = \binom{n}{2} p(1-p)$ $\quad$ poly $|n|$ what degree

$\boxed{\text{HW}}$ $T := \#\text{triangles}$ $\quad$ (a) $E(T)$ $\quad$ (b) $\mathrm{Var}(T)$

exact expression, asymptotics $a \cdot n^b$