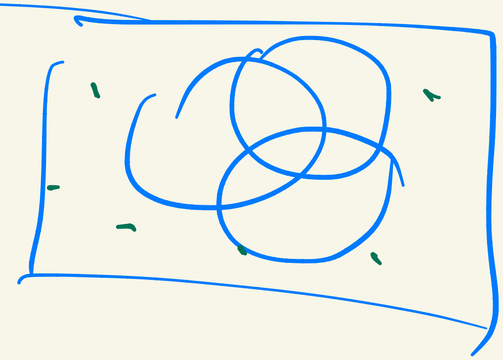
INCLUSION-EXCLUSION

$$P(A\cup B)=P(A)+P(B)-P(A\cap B)$$

$$P(\overline{A\cup B})=\underbrace{P(\Omega)}_{1}-P(A)-P(B)+P(A\cap B)$$

INCLUSION – EXCLUSION for 3 events

$$P(\overline{A \cup B \cup C}) = P(\Omega) - P(A) - P(B) - P(C)$$
$$+ P(A \cap B) + P(A \cap C) + P(B \cap C)$$
$$- P(A \cap B \cap C)$$



INCLUSION-EXCLUSION Formula

Let $A_1 \dots A_k \subseteq \Omega$ be events, $B = \bigcup_{i=1}^{k} A_i$

Then

$$P(B) = S_0 - S_1 + S_2 - + \cdots + (-1)^k S_k \quad \leftarrow 2^k \text{ terms}$$

where

$$S_i = \sum_{1 \le l_1 < l_2 < \cdots < l_i \le k} (-1)^i P(A_{l_1} \cap \cdots \cap A_{l_i}) \quad \leftarrow \binom{k}{i} \text{ terms}$$

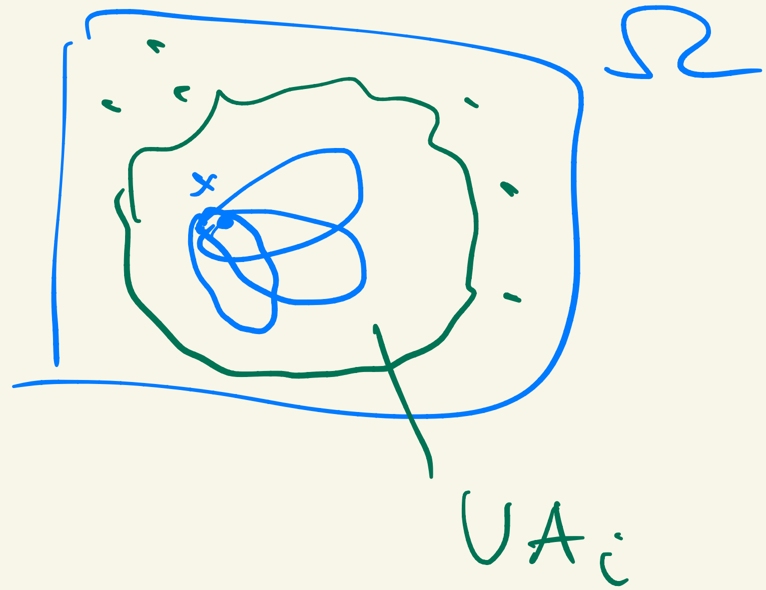
	# terms
$S_0 = P(\Omega) = 1$	1
$S_1 = \sum P(A_i)$	k
$S_2 = \sum_{1 \le i < j \le k} P(A_i \cap A_j)$	$\binom{k}{2}$
$S_3 = \sum_{1 \le l_1 < l_2 < l_3 \le k} P(A_{l_1} \cap A_{l_2} \cap A_{l_3})$	$\binom{k}{3}$

$\vdots$

Proof degree of $x \in \Omega$

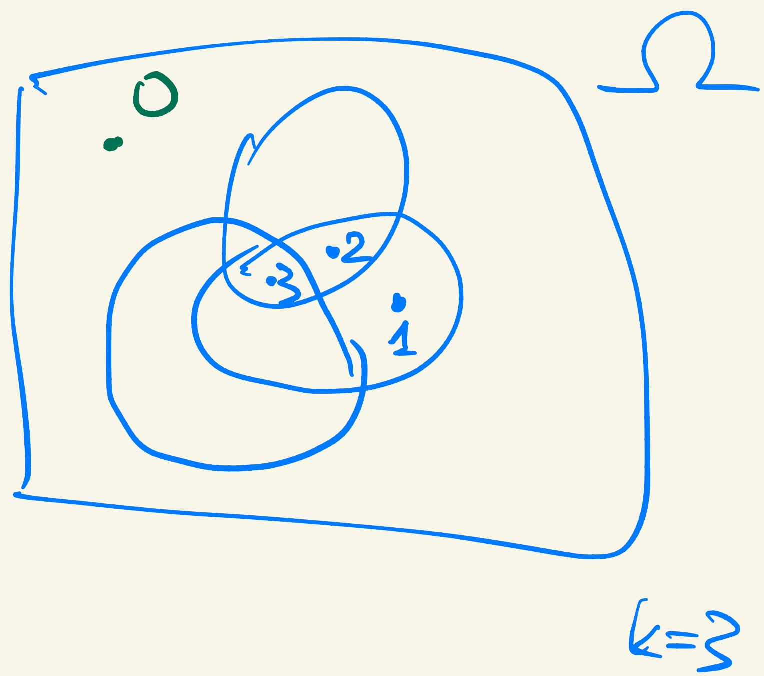
$$\deg(x) := \left|\{i \mid x \in A_i\}\right|$$

$$B = \{x \in \Omega \mid \deg(x) = 0\}$$



QUESTION # times $x \in \Omega$ is counted in I/E fmla?

Goal $\begin{cases} 1 & \text{if } \deg(x) = 0 \\ 0 & \text{times } \deg(x) > 0 \end{cases}$



Let $\deg(x) = r$

# times counted in	
S_0	1
S_1	r
S_2	$\binom{r}{2}$
$\vdots$	
S_i	$\binom{r}{i}$

$$P(B) = S_0 - S_1 + S_2 - S_3 + - \cdots$$

x is counted

$$1 - r + \binom{r}{2} - \binom{r}{3} + \cdots \quad \text{times}$$

$$\binom{r}{0} - \binom{r}{1} + \binom{r}{2} - \binom{r}{3} + - \cdots = (1-1)^r$$

$$= 0^r = \begin{cases} 0 & \text{if } r \neq 0 \\ 1 & \text{if } r = 0 \end{cases}$$

✓

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k \quad \leftarrow \text{ Binomial Theorem}$$

2nd proof

LEMMA $$\prod_{i=1}^{k}(1+x_i) = \sum_{I\subseteq[k]} \prod_{i\in I} x_i$$

Note $\prod_{i\in\emptyset} x_i = 1$

$$(1+x_1)(1+x_2)(1+x_3) = 1+x_1+x_2+x_3+x_1x_2+x_1x_3+x_2x_3+x_1x_2x_3$$

$$(1-x_1)(1-x_2)(1-x_3) = 1-x_1-x_2-x_3+x_1x_2+x_1x_3+x_2x_3-x_1x_2x_3$$

$$\prod(1-x_i) = \sum_{I\subseteq[k]}(-1)^{|I|}\prod_{i\in I} x_i$$

Inclusion-Exclusion restated

$$P(B) = S_0 - S_1 + S_2 - S_3 + \cdots$$

$$= \sum_{I \subseteq [k]} (-1)^{|I|} P\left(\bigcap_{i \in I} A_i\right)$$

$A \subseteq \Omega$ indicator of A : $\vartheta_A : \Omega \to \mathbb{R}$

for $x \in \Omega$ $\quad \vartheta_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$

$$\boxed{E(\vartheta_A) = P(A)}$$

$$\boxed{\vartheta_{A\cap B} = \vartheta_A \cdot \vartheta_B}$$

$$x \in A\cap B \iff x\in A \wedge x \in B$$

$$\boxed{\vartheta_{\overline{A}} = 1 - \vartheta_A}$$

$$x \in \overline{A} \iff x \notin A$$

$$\boxed{\vartheta_{A_1\cap\cdots\cap A_j} = \prod_{i=1}^{j} \vartheta_{A_i}}$$

INCLUSION-EXCLUSION

$$B = \overline{A_1 \cup \cdots \cup A_k} = \overline{A_1} \cap \cdots \cap \overline{A_k}$$

$$\vartheta_B = \prod_{i=1}^{k} \vartheta_{\overline{A_i}} = \prod_{i=1}^{k}(1-\vartheta_{A_i}) = \sum_{I\subseteq[k]} (-1)^{|I|} \cdot \prod_{i\in I} \vartheta_{A_i}$$

INCLUSION-EXCLUSION

$$B = \overline{A_1 \cup \dots \cup A_k} = \overline{A_1} \cap \dots \cap \overline{A_k}$$

$$\vartheta_B = \prod_{i=1}^{k} \vartheta_{\overline{A_i}} = \prod_{i=1}^{k} (1-\vartheta_{A_i}) = \sum_{I \subseteq [k]} (-1)^{|I|} \cdot \prod_{i \in I} \vartheta_{A_i}$$

$$= \sum_{I \subseteq [k]} (-1)^{|I|} \cdot \vartheta_{\bigcap_{i \in I} A_i}$$

$$\boxed{P(B) = E(\vartheta_B)} = E(\text{—}\,''\,\text{—}) = \sum_{I \subseteq [k]} (-1)^{|I|} E(\vartheta_{\bigcap_{i \in I} A_i})$$

$$\boxed{= \sum_{I \subseteq [k]} (-1)^{|I|} \cdot P(\bigcap_{i \in I} A_i)}$$

✓

Application

Count $f : [n] \to [k]$ surjections $(\mathrm{range}(f) = [k])$

$$A_i := \{ f \in [k]^{[n]} \mid i \notin \mathrm{range}(f) \} \qquad i \in [k]$$

$$\boxed{\begin{array}{l} \Omega = [k]^{[n]} \\ \text{unif. distr.} \end{array}}$$

$$B = \overline{A_1 \cup \dots \cup A_k}$$

$$P(B) = \frac{\#\text{surj.}}{k^n} \qquad \#\text{surj.} = k^n \, P(B)$$

$$P(B) = I/E$$

$$P\Big(\bigcap_{i \in I} A_i\Big)$$

$$I \subseteq [k] \qquad \Big|\bigcap_{i \in I} A_i\Big| = (k - |I|)^n$$

$$= 1 - k \cdot \frac{(k-1)^n}{k^n} + \binom{k}{2} \cdot \frac{(k-2)^n}{k^n} - \binom{k}{3} \frac{(k-3)^n}{k^n} + - \dots$$

$$\#\text{surj.} = k^n - k \cdot (k-1)^n + \binom{k}{2} \cdot (k-2)^n - \binom{k}{3} (k-3)^n + - \dots$$