PROBLEM SESSION

2023-12-01

1

16.27 (a) $\alpha(G) = 1 \iff G \cong K_n$

$*$ $1 \leq \alpha(G) \leq n$

Pf

$(\forall v \in V)(\{v\} \text{ indep})$

A indep: $A \subseteq V$

i.e.

$\alpha(G) \neq 1 \iff G \not\cong K_n$

$*$ $\Updownarrow$

$\alpha(G) \geq 2$

$\Updownarrow$

$\exists S \subseteq V,\ |S| = 2,$ indep.

$\Updownarrow$

$(\exists u \neq v)(u \nsim v)$

$\Updownarrow$

G is not complete

16.27(b) $\alpha(G) = n \overset{?}{\iff} G \cong \overline{K_n}$

$\Updownarrow$

$\exists$ indep set $S \subseteq V$
of size n

$\Updownarrow$

V is indep

Greedy indep. set algorithm

A: potential indep set

```
    A := ∅                      // initialization
(*) for v ∈ V                   // V is an ordered set
      for w ∈ A
        if v ~ w then
          goto (*)              // next v            ] explore v
      end(for)
      A := A ∪ {v}              // add v to A
    end(for)
    return A
```

if $(\forall w \in A)(w \not\sim v)$ then

Proof of correctness

Lemma

$A_0 \subseteq A_1 \subseteq \dots \subseteq A_n$ ✓

① <u>Claim</u> A is independent throughout the algorithm

<u>Pf</u> Steps $0, 1, \dots, n$

$A_i :=$ "A" after Step $\#i$

$A_0 = \emptyset$ (initialization) indep ✓

Now $i \geq 1$

IH A_{i-1} indep

DC A_i indep

<u>Pf</u> if $A_i = A_{i-1}$ ✓

o/w $A_i = A_{i-1} \cup \{v_i\}$

but $(\forall w \in A_{i-1})(v_i \not\sim w)$ b/c lines 3-6 ✓

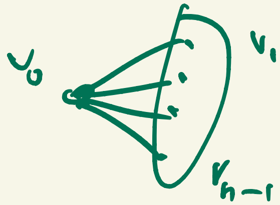
② Claim A is <u>maximal</u> indep

Pf. Suppose $v_i \notin A$ NTS $A \cup \{v_i\}$ <u>not</u> indep

Pf. $v_i \notin A \Rightarrow v_i \notin A_i$ why? b/c lines 3-5 <u>exited</u> in round $\#i$ b/c $v_i \sim w$ $\exists w \in A_{i-1} \subseteq A$

$\therefore \{v_i\} \cup A$ not indep

16.29 (a) find conn. G s.t. $\alpha(G) = n-1$



— we construct G

Pf of correctness: ① $\alpha(G) < n$ b/c $G \not\cong \overline{K_n}$

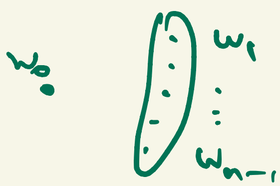
② $\alpha(G) \geq n-1$ b/c $\{v_1 \dots v_{n-1}\}$ indep

(b) This example is the only example up to isomorphism

i.e. If H conn., $\alpha(H) = n-1$ then $H \cong G$

Pf H ← given to us by adversary

$\alpha(H) = n-1$ $\therefore$ $\exists S \subseteq V$, $|S| = n-1$, indep



$\{w_1 \cdots w_{n-1}\}$ indep.

Claim $(\forall 1 \leq i \leq n-1)(w_0 \sim w_i)$

Pf: otherwise we had $\deg(w_i) = 0$ isolated → ← H is conn.

Let $f(w_i) = v_i$ this is an isomorphism (check!) ✓ ↖ We looked at each pair of vertices

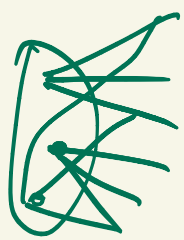
$\therefore$ $G \cong H$

16.39 If G regular and $E \neq \emptyset$
then $\alpha(G) \leq \lfloor \frac{n}{2} \rfloor$

Pf. reg of degree d $\therefore$ $m = |E| = \frac{dn}{2}$ | Handshake Thm

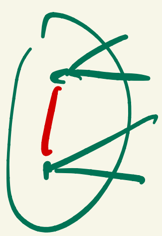
Let $A \subseteq V$ be indep.

A



all the $d \cdot |A|$ edges leaving A are distinct b/c A is indep

$\therefore$ $m \geq d \cdot |A|$



$\therefore$ $d|A| \leq m \leq d \cdot \frac{n}{2}$

$\therefore$ $|A| \leq \frac{n}{2}$ ✓

$\therefore$ $|A| \leq \lfloor \frac{n}{2} \rfloor$

16.41 (a) $\alpha(G)\cdot\chi(G)\geq n$

Pf $k := \chi(G)$ take a k-coloring of G: $f: V \to [k]$

color classes: $V_1 \cdots V_k$ they partition V

$\therefore n = |V| = |V_1| + \cdots + |V_k|$ $\qquad V_i = f^{-1}(i)$

each V_i is indep by def of legal coloring

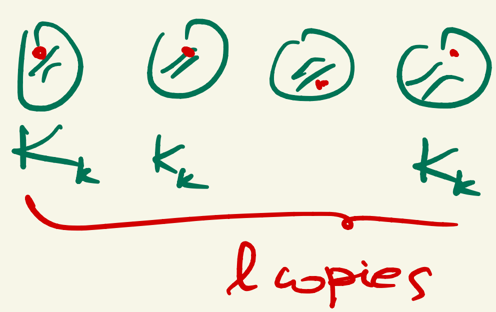
$\therefore |V_i| \leq \alpha(G)$

$\therefore V = \sum_{i=1}^{k} |V_i| \leq \sum_{i=1}^{k} \alpha(G) = k\cdot\alpha(G) = \alpha(G)\cdot\chi(G)$

(b) tight: if $n = k\cdot\ell$ then $(\exists G)(\chi(G) = k,\ \alpha(G) = \ell)$

Pf $G = \ell\cdot K_k$

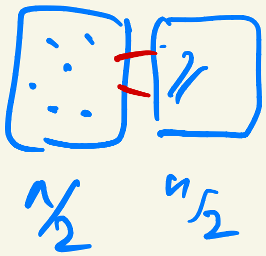
$\chi(K_k) = k$	$\alpha(G) \geq \ell$
$\therefore \chi(G) = k$	$\alpha(G) \leq \ell$ PHP
	NTS $\forall$ indep set has size $\leq \ell$



Lemma
$G = H \dot\cup L$
then
$\chi(G) = \max\{\chi(H), \chi(L)\}$

$\ominus$ $\oplus$

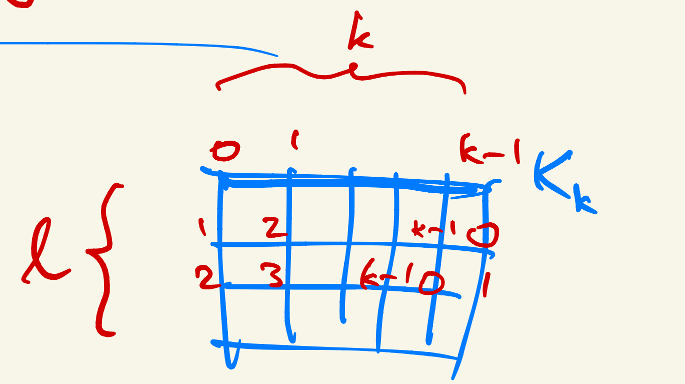
(c) Find G_n s.t. $\alpha(G_n) \cdot \chi(G_n) = \Omega(n^2)$



wrong if $k < \ell$

? WLOG

$G = K_k \square K_\ell$ works for $k \geq \ell$



$\chi(G) \geq k$ b/c $G \supseteq K_k$

$\chi(G) \leq k$ NTS G is k-colorable

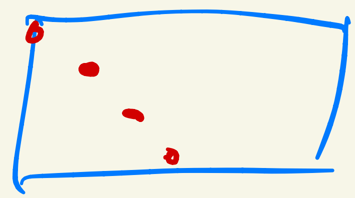
need to find a k-coloring

$V = \{0, \dots, k-1\} \times \{0, \dots, \ell-1\}$

$c(i,j) = i + j \mod k$

$\alpha(G) \leq \ell$ NTS $\forall$ indep set has size $\leq \ell$

PHP pigeonholes: rows



$\alpha(G) \geq \ell$

need $\ell \leq k$: diagonal is independent

9

16.71 UNION BOUND $P\left(\bigcup_{i=1}^{k} A_i\right) \le \sum_{i=1}^{k} P(A_i)$

Pf 1 induction on k

$k=2$ $\quad P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2) \le$

$\le$ — " —

Assume $\boxed{k \ge 3}$

Ind. Hyp: true for all $k' < k$

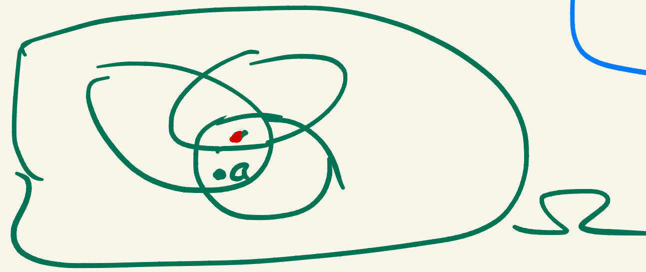
Pf for k: Let $B = A_1 \cup \dots \cup A_{k-1}$

$$P\left(\bigcup_{i=1}^{k} A_i\right) = P(B \cup A_k) \le P(B) + P(A_k) \le \sum_{i=1}^{k-1} P(A_i) + P(A_k) = \sum_{i=1}^{k} P(A_i)$$

IH $k'=2$ $\qquad$ I.H. $k' = k-1$

✓

UNION BOUND



Pf #2

$$P(A_i) = \sum_{a \in A_i} P(a)$$

$$\sum_{i=1}^{k} P(A_i) = \sum_{i=1}^{k} \sum_{a \in A_i} P(a) = \sum_{a \in \Omega} P(a) \cdot \deg(a) =$$

$$\deg(a) = |\{i \mid a \in A_i\}|$$

$$\sum_{\deg(a) \geq 1} P(a) \cdot \deg(a) \geq \sum_{\substack{a \in \Omega \\ \deg(a) \geq 1}} P(a) = P\left(\bigcup_{i=1}^{k} A_i\right)$$ ✓

$$(\forall a \in \Omega)\left(\deg(a) \geq 1 \iff a \in \bigcup_{i=1}^{k} A_i\right)$$

16.49

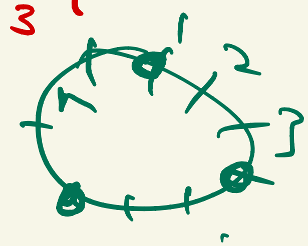
DEF $\beta(G)$ = min size of maximal indep. sets

$$\beta(C_n) = \left\lceil \frac{n}{3} \right\rceil$$

① $\beta(C_n) \le \left\lceil \frac{n}{3} \right\rceil$

NTS: $\exists$ maximal indep set of size $\le \left\lceil \frac{n}{3} \right\rceil$

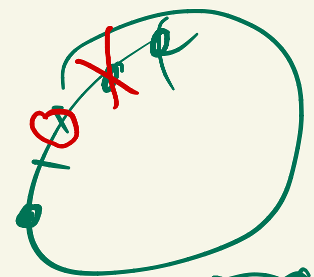
Lazy alg. finds it



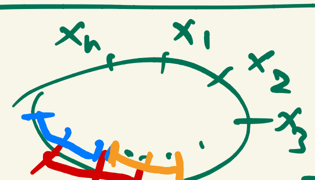
② $\beta(C_n) \ge \left\lceil \frac{n}{3} \right\rceil$

NTS: $\forall$ maximal indep set has size $\ge \frac{n}{3}$

Pf 1: every interval def'd by A has ≤ 3 edges $\therefore$ # intervals $\ge \frac{n}{3}$



Pf 2:

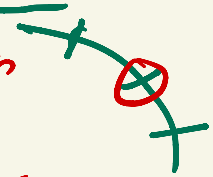


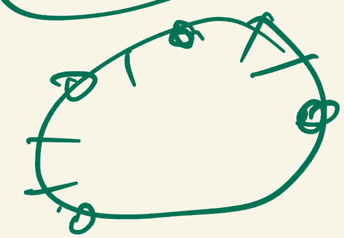
$x_i = \begin{cases} 1 & \text{if } i \in A \\ 0 & \text{o/w} \end{cases}$ $\quad |A| = \sum x_i$

$x_i + x_{i+1} + x_{i+2} \ge 1$ indices mod n

$3 \cdot \sum x_i \ge n$

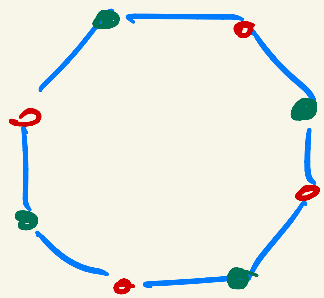
n inequalities add up all





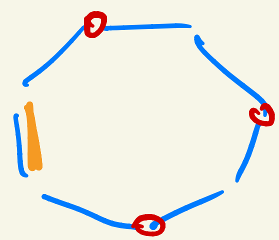
16.51 (a) Count maximum indep sets in C_n $\qquad \alpha(C_n) = \left\lfloor \frac{n}{2} \right\rfloor$

n even: $\alpha = \frac{n}{2}$



$\# = 2$

n odd: $\alpha = \frac{n-1}{2}$



every maximum indep set misses an edge

$\# = n$

bijection between edges and maximum indep sets

(b) $O(n)$? Y $\qquad 2 = O(n) \qquad n = O(n)$ $\qquad$ implied const. $c = 1$

$\Theta(n)$? N $\qquad$ b/c not $\Omega(n)$

b/c n is not $O(2)$

16.65

$B = \{0,1\}$

B^n = all Boolean strings of length n

$s \in B^n$

$k(0101110) = 4$

DEF $f(s) := p^{k(s)}(1-p)^{n-k(s)}$ where $\underbrace{k(s)}_{\text{weight}} = \#\ 1\text{s in } s$

Claim f is a prob distrib on B^n

$|B^n| = 2^n$

1. $(\forall s \in B^n)(f(s) \geq 0)$ ✓
2. NTS $\displaystyle\sum_{s \in B^n} f(s) = 1$

$|\{s \in B^n \mid k(s) = t\}| = \binom{n}{t}$

$$\therefore \sum_{s \in B^n} f(s) = \sum_{t=0}^{n} \underbrace{\sum_{\substack{s \in B^n \\ k(s)=t}} f(s)}_{p^t(1-p)^{n-t}} = \sum_{t=0}^{n} \binom{n}{t} p^t (1-p)^{n-t} = (p + (1-p))^n = 1^n$$

$= 1$ ✓

16.67 # trivial events is a power of 2

$$\boxed{P(\bar{D}) = 1 - P(D)}$$

	$\emptyset$	Ω
Pr	0	1

Let $A = \{x \in \Omega \mid P(x) = 0\}$ $\qquad k := |A|$

Let $B \subseteq \Omega$.

$$P(B) = 0 \iff B \subseteq A$$

$\therefore$ # events of prob zero is 2^k $= |\mathcal{P}(A)|$

$C \subseteq \Omega$

$$[P(C) = 1 \iff C \supseteq \bar{A}]$$

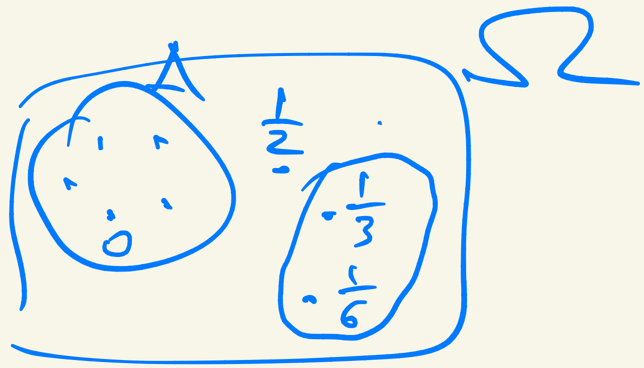
$$P(C) = 1 \iff P(\bar{C}) = 0$$

bijection between events of ~~prob~~ 1 and " of prob 0

$$C \mapsto \bar{C}$$

$\therefore$ # events of prob 1 is 2^k

trivial events is $2^k + 2^k = 2^{k+1}$



17.29 3 dice

17.33 prob of causes (W, unif.)

W = population of patient with the symptom

A : suffers from disease A

B : B

C : something else

$\{A, B, C\}$

$\downarrow$

0.7 . . .

17.37 independence of complement

ASSN A, B indep

DC $\bar{A}, B$ indep

$P(A\cap B) = P(A)\cdot P(B)$

$\underbrace{P(\bar{A}\cap B)}$ $\stackrel{?}{=} P(\bar{A})\cdot P(B)$

$\stackrel{?}{=} (1-P(A))\cdot P(B)$

✓ $B = (A\cap B)\,\dot{\cup}\,(\bar{A}\cap B)$

$P(B) = P(A\cap B) + P(\bar{A}\cap B)$

disjoint b/c

$A\cap B \subseteq A$

$\bar{A}\cap B \subseteq \bar{A}$] disj

$\underline{P(B) - P(A\cap B)} \stackrel{?}{=} (1-P(A))P(B) =$

$\underline{= P(B) - P(A)\cdot P(B)}$

they are equal $\Leftrightarrow P(A\cap B) = P(A)\cdot P(B)$ ✓

true by ASSN.

b/c distributivity:

$(A\cap B)\cup(\bar{A}\cap B) = (A\cup\bar{A})\cap B = \underline{\Omega\cap B = B}$

17.41 die < odd $\frac{1}{2}$, square $\frac{1}{3}$ } both: $\frac{1}{6}$ So indep ✓

17.43 $|\Omega| = p$ uniform

$\Rightarrow$ no nontriv indep events

17.49 Suppose $A \subseteq B$ and A, B indep. Then A or B triv.

Pf $A \subseteq B \iff A = A \cap B$

$$P(A) = P(A \cap B) = P(A) \cdot P(B)$$

$$P(A)\big(1 - P(B)\big) = 0$$

$$\therefore \quad P(A) = 0 \quad \text{or} \quad 1 - P(B) = 0$$

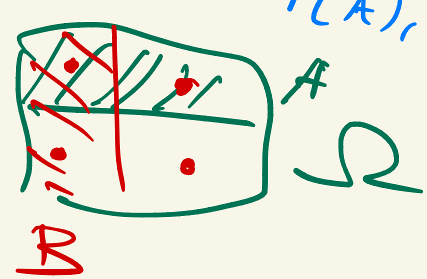
17.45 A, B nontriv. indep $\Rightarrow |\Omega| \geq 4$

Pf
A, B indep $\Rightarrow P(A \cap B) = P(A)P(B) > 0$ $\quad 0 < P(A), P(B) < 1$

$\bar{A}, B$ " $\quad P(\bar{A} \cap B) > 0$ $\quad 0 < P(\bar{A}), P(\bar{B}) < 1$

$A, \bar{B}$ " $\quad P(A \cap \bar{B}) > 0$

$\bar{A}, \bar{B}$ " $\quad P(\bar{A} \cap \bar{B}) > 0$



These 4 intersections are disjoint

E.g. $(\bar{A} \cap \bar{B}) \cap (\bar{A} \cap B) \subseteq \bar{B} \cap B = \emptyset$

17.51 if $A \cup B$ and $A \cap B$ are indep $\overset{?}{\not\Rightarrow}$ A or B triv

NO Pf counterexample $\Omega = \{1, 2\}$ unif $A = \{1\}$ $B = \{2\}$

$A \cap B = \emptyset$, A, B nontriv

17.61 If A, B, C indep then $A \cup B$ and C indep

17.101 $\min X \leq E(X) \leq \max X$

$$E(X) = \sum_{a \in \Omega} X(a) \cdot P(a) \underset{b/c\ P(a) \geq 0}{\geq} \sum_{a \in \Omega} (\min X) \cdot P(a) = \min X \cdot \underbrace{\sum_{a \in \Omega} P(a)}_{1} = \min X$$

$E(X) \leq \max X$ — same argument

OR $E(X) = -E(-X) \leq \max X$

$E(-X) \geq \min(-X) = -\max X$

17.117 envelopes randomly stuffed

$\Omega := \{\text{all permutations of the envelopes}\}$

$$\vartheta_i = \begin{cases} 1 & \text{if } i^{th} \text{ letter goes in right envelope} \\ 0 & \text{o/w} \end{cases}$$

$X = \sum \vartheta_i \quad = \#$ letters that go into right envelope

$$E(X) = \sum E(\vartheta_i) = \sum P(\underbrace{\vartheta_i = 1}_{i^{th} \text{ letter goes into right env.}}) = n \cdot \frac{1}{n} = \underline{\underline{1}}$$

by symmetry $P(\vartheta_i = 1) = \frac{1}{n}$

2nd proof that $P(\vartheta_i = 1) = \frac{1}{n}$:

$$\underline{\underline{L}} \; \frac{(n-1)!}{n!} = \frac{1}{n}$$

17.113 E # Aces, # Spades in poker hand

ϑ_i : indicator that i^{th} card is an Ace / Spade

$i = 1, \dots, 5$

$$X = \#\text{Aces} / \text{Spades} = \sum_{i=1}^{5} \vartheta_i$$

$$E(X) = \sum_{i=1}^{5} E(\vartheta_i) = \sum P(\vartheta_i = 1)$$

by symmetry

$$P(i^{th}\text{ card Ace}) = \frac{4}{52} = \frac{1}{13}$$

$$P(i^{th}\text{ card Spade}) = \frac{13}{52} = \frac{1}{4}$$

$$\therefore \quad E(\#\text{Aces}) = \frac{5}{13}$$

$$E(\#\text{Spades}) = \frac{5}{4}$$

$$\cancel{|\Omega| = \binom{52}{5}}$$ not for this proof

$$|\Omega| = 52 \cdot 51 \cdot 50 \cdot 49 \cdot 48$$

$$|\Omega| = \binom{52}{5}$$

$j \in \{\spadesuit \ \heartsuit \ \diamondsuit \ \clubsuit\}$

Y_j = indicator that Ace of suit j is in the hand

$j = 1, \dots, 4$

$$X = \sum_{j=1}^{4} Y_j$$

$$E(X) = \sum_{j=1}^{4} E(Y_j) = \sum_{j=1}^{4} P(Y_j = 1) = \underbrace{4}_{\text{4 suits}} \cdot \frac{5}{52} = \frac{5}{13}$$

$$\frac{\binom{51}{4}}{\binom{52}{5}} = \frac{5}{52}$$

Same for Spades

$$\sum_{i=1}^{13} \frac{5}{52} = \underbrace{13}_{\text{13 kinds}} \cdot \frac{5}{52} = \frac{5}{4}$$

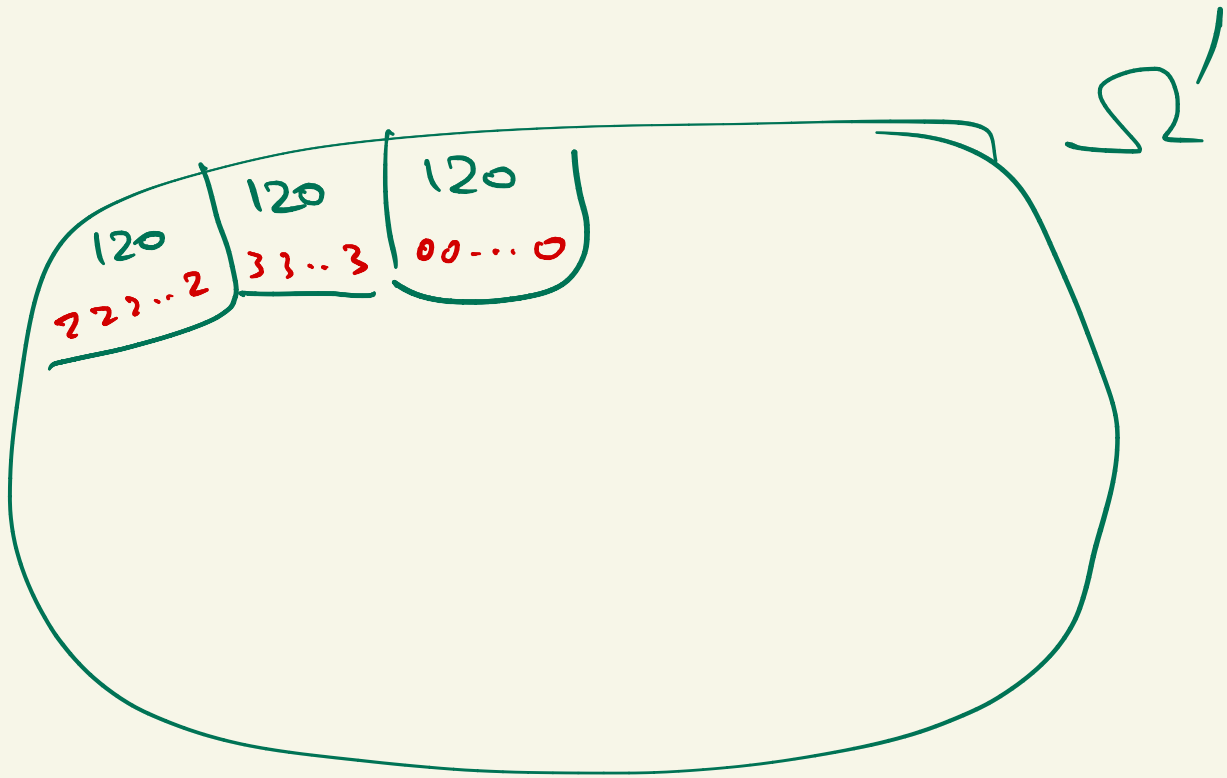
Compare the two sample spaces

$|\Omega| = \binom{52}{5}$ set of poker hands

$|\Omega'| = 5!\binom{52}{5}$ " <u>ordered</u> poker hand

<u>equiv. rel. on Ω':</u>

same unordered set



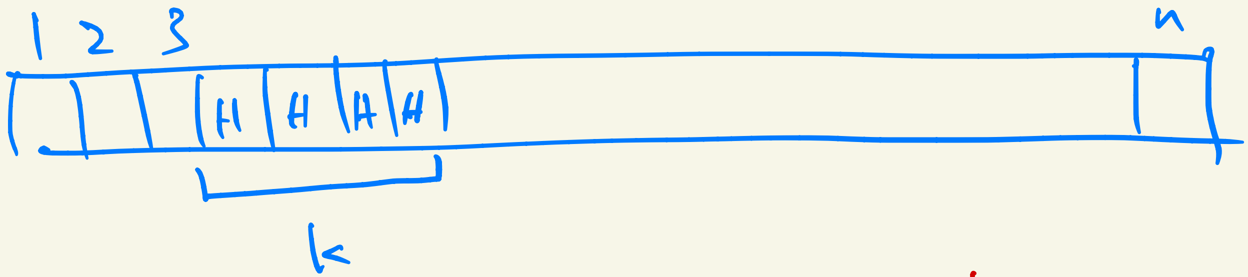
$$|\Omega| = \frac{1}{120}\cdot|\Omega'|$$

$X : \Omega' \to \mathbb{N}_0$

X is constant on each equiv. class

17.115 $E(\#\text{runs of } k \text{ heads})$

$P(\text{heads}) = p$



$$P(x_i = x_{i+1} = \dots = x_{i+k-1} = H) = p^k$$

X = # runs of k heads

Y_i = indicator that a run of k heads starts at coin #i

$$X = \sum_{i=1}^{n-k+1} Y_i$$

$$E(X) = \sum_{i=1}^{n-k+1} E(Y_i) = \sum_{i=1}^{n-k+1} P(Y_i = 1) = \underline{\underline{(n-k+1)\cdot p^k}}$$

17.119 Club of 2000

ϑ_i = indicator that i^{th} member is lucky

X = # lucky members $= \sum \vartheta_i$

$$E(X) = \sum_{i=1}^{2000} E(\vartheta_i) = \sum_{i=1}^{2000} \underbrace{P(i^{th} \text{ member is lucky})}_{\text{by symmetry } \frac{1}{2000}} = 2000 \cdot \frac{1}{2000} = \underline{\underline{1}}$$

assuming

member is was born in $[2000]$

$$|\Omega| = 2000!$$

2^{nd} solution:

ψ_i = indicator that i^{th} card goes to lucky member

$$X = \sum_{i=1}^{2000} \psi_i$$

$$E(X) = \sum E(\psi_i) = \sum P(\psi_i = 1)$$

$$P(i^{th} \text{ card goes to lucky member}) = \frac{c_i}{2000}$$

where c_i = # members born in year i

$$\therefore E(X) = \sum \frac{c_i}{2000} = \frac{1}{2000} \underbrace{\sum c_i}_{2000} = 1$$

b/c vodka

15.155 $G \not\supset K_3$ $\quad u \sim v$

$$\Longrightarrow \deg u + \deg v \le n$$

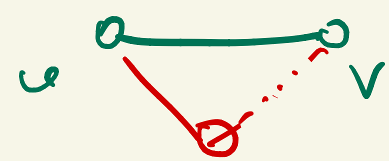
Pf $N(u) = \{w \in V \mid w \sim u\}$ set of neighbors of u

$$|N(u)| + |N(v)| = \underbrace{|N(u) \cup N(v)|}_{\subseteq V} + \underbrace{|N(u) \cap N(v)|}_{0} \le n$$

this is tight:

$K_{t,t}$

$\deg(u) = \deg(v) = t = \frac{n}{2}$



$$\frac{1+4+5+2}{4} = 3$$

$$\frac{\overbrace{1+\cdots+1}^{120}+\overbrace{4+\cdots+4}^{120}+\overbrace{5+\cdots+5}^{120}+\overbrace{2+\cdots+2}^{120}}{120\cdot 4} = 3$$