PROBLEM SESSION

2023-12-07

18.35 X : # Aces
Y : # Spades in poker hand

X, Y not independent

$P(X=4) > 0$

$P(Y=0) > 0$

$P(X=4 \wedge Y=0) = 0 \neq P(X=4) \cdot P(Y=0)$

18.75 $G(n,p)$ model: prob. distr. on the $2^{\binom{n}{2}}$ graphs with $V=[n]$

$(\forall i\neq j)(P(i\sim j)=p)$

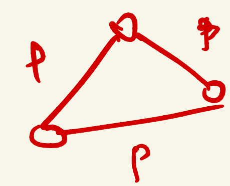
$i,j\in[n]$

↑ these $\binom{n}{2}$ events are indep.

indicator the i^{th} triangle of K_n is present in G

$$X := \#\text{ triangles in } G = \sum_{i=1}^{\binom{n}{3}} Y_i$$

$$E(X)=\sum E(Y_i) = \sum \underbrace{P(i^{th}\text{ triangle} \subseteq G)}_{p^3} = \binom{n}{3}p^3$$

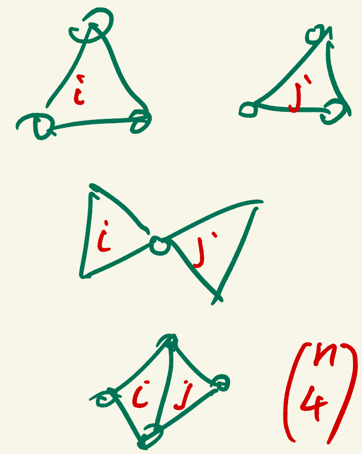


$$\mathrm{Var}\, X = \sum_i\sum_j \underline{\mathrm{Cov}(Y_i,Y_j)}$$

$\mathrm{Cov}(Y_i,Y_j)=\mathrm{Cov}(Y_j,Y_i)$

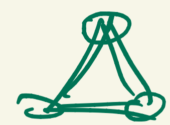
$$\operatorname{Var} X = \sum_i \sum_j \operatorname{Cov}(Y_i, Y_j)$$

$\operatorname{Cov} = 0$



$$\operatorname{Cov}(Y_i, Y_j) = \underbrace{E(Y_i Y_j)}_{p^5} - \underbrace{E(Y_i)}_{p^3}\underbrace{E(Y_j)}_{p^3} = p^5 - p^6 = p^5(1-p)$$

$$\binom{n}{4}\cdot 4 \cdot 3 = \frac{n(n-1)(n-2)(n-3)}{2}$$



$i = j$

$\binom{n}{3}$

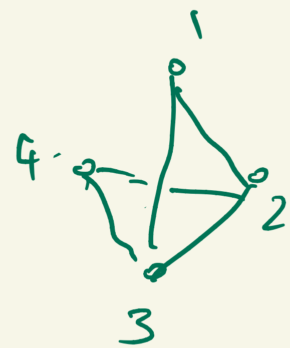
$$\operatorname{Cov}(Y_i, Y_i) = p^3 - p^6 = p^3(1-p^3)$$

$$\operatorname{Var} X_n = \binom{n}{3} p^3(1-p^3) + 12\cdot\binom{n}{4}\cdot p^5(1-p)$$

fix p $\quad 0 < p < 1$

$n \to \infty$

$$\operatorname{Var} X_n \sim \frac{n^4}{2}\cdot p^5(1-p)$$

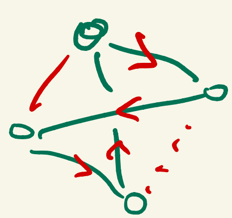


$(1, 2, 3, 4)$

$2 \leftrightarrow 3$

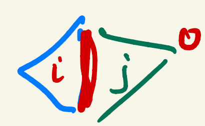
$(1\ 3\ 2\ 4)$

$$\frac{3\cdot 2}{2}\cdot 2 \cdot 2$$



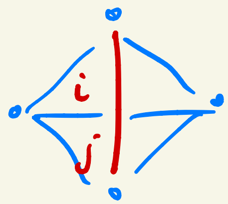
$$\binom{n}{3}\cdot 3 \cdot (n-3) = \frac{n(n-1)(n-2)(n-3)}{2}$$

i



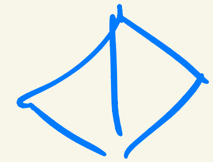
$n-3$ choices

$\binom{n}{4} \cdot 6 \cdot 2$ choices for (y_i, y_j)



K_4^-

6 choices for edge removed



K_4^-

5

18.39 $X = \#$ Aces $Y = \#$ Spades uncorrelated

$E(XY) = E(X)E(Y)$

rs cards: r Aces (r suits)
s Spades (s kinds)

in standard deck:
$r = 4$
$s = 13$
$k = 5$ poker

hand: k cards

$|\Omega| = \binom{rs}{k} \cdot k!$ ← # ordered hands

A_i : "i^{th} card Ace"
B_j : "j^{th} card Spade"

$i, j \in [k]$

$P(A_i) = \frac{r}{rs} = \frac{1}{s}$ $P(B_j) = \frac{s}{rs} = \frac{1}{r}$

$E(X) = \sum P(A_i) = \frac{k}{s}$

$E(Y) = \sum P(B_j) = \frac{k}{r}$

$E(X) \cdot E(Y) = \frac{k^2}{rs}$

X_i : ind A_i $\quad X = \sum X_i$

Y_j : ind B_j $\quad Y = \sum Y_j$

$$XY = \sum X_i \cdot \sum Y_j = \sum_i \sum_j X_i Y_j$$

$$E(XY) = \sum\sum E(\underbrace{X_i Y_j}) = \sum\sum P(A_i \cap B_j)$$

ind "ith card Ace
jth card Spade"

$A_i \cap B_i$: ith card is Ace of Spades

<u>Case</u> $i = j$ $\quad P(A_i \cap B_i) = \frac{1}{rs}$

<u>Case</u> $i \neq j$

(a) ith card: Ace of Spades $\quad \frac{1}{rs} \cdot \frac{s-1}{rs-1}$

$s-1$ ← # spades remaining

$rs-1$ ← # cards remaining

$A_i \cap B_i$

$\frac{s-1}{rs-1}$ ← $P(B_j \mid A_i \cap B_i)$

(b) ith card: Ace, not Spades $\quad \frac{r-1}{rs} \cdot \frac{s}{rs-1}$

$A_i \cap \overline{B_i}$

$\frac{s}{rs-1}$ ← $P(B_j \mid A_i \cap \overline{B_i})$

$$P(A_i \cap B_j) = \frac{s-1+(r-1)s}{rs(rs-1)} = \frac{1}{rs}$$

$$\therefore E(XY) = \frac{k^2}{rs}$$

k^2 ← # pairs (i,j)

$\therefore \forall i,j \quad$ <u>A_i, B_j indep</u>

19.25 n letters placed in n envelopes at random

$|\Omega| = n!$

$$P(\underbrace{\text{all letters misplaced}}_{B}) = S_0 - S_1 + S_2 - + \cdots$$

A_i: i^{th} letter goes in right envelope $\quad P(A_i) = \frac{1}{n}$ by symm.

$$S_j = \underbrace{\sum_{\substack{I \subseteq [n] \\ |I| = j}}}_{\#\text{terms } \binom{n}{j}} P\left(\bigcap_{i \in I} A_i\right) = \binom{n}{j} \cdot \frac{(n-j)!}{n!} = \frac{1}{j!}$$

$$= \sum_{j=0}^{n} (-1)^j \cdot \frac{1}{j!} \xrightarrow[n \to \infty]{} \frac{1}{e}$$

$$\frac{1}{e} = \sum_{j=0}^{\infty} (-1)^j \frac{1}{j!}$$

19.23 # n-digit integers s.t.
all digits are odd
all odd digits occur

digits $f(1), \dots, f(n)$ $\qquad f : [n] \to \{1,3,5,7,9\}$ surj.

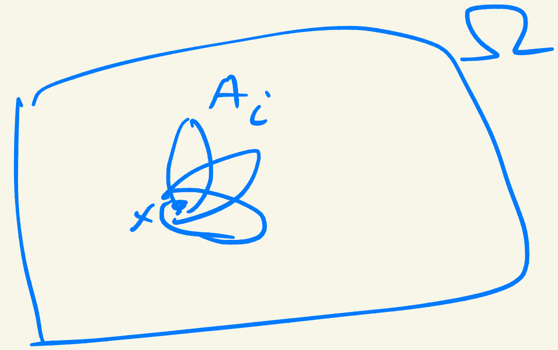
surjections: $5^n - \binom{5}{1} \cdot 4^n + \binom{5}{2} \cdot 3^n - \binom{5}{3} \cdot 2^n + \binom{5}{4} \cdot 1^n$

19.21 Bonferroni's inequalities

$$
\begin{aligned}
P(B) &\le S_0 = 1 \\
&\ge S_0 - S_1 \\
&\le S_0 - S_1 + S_2 \\
&\ge S_0 - S_1 + S_2 - S_3 \\
&\vdots
\end{aligned}
$$

Assume t is odd

$$P(B) \geq S_0 - S_1 + - \cdots - S_t$$



$$r := \deg(x) := \left|\{ i \mid x \in A_i \}\right|$$

x was counted $T(r) = \binom{r}{0} - \binom{r}{1} + - \cdots - \binom{r}{t}$ times

if $r = 0$ $\quad T(0) = 1$

it suffices to show: if $r \geq 1$ then $T(r) \leq 0$

NTS: $\binom{r}{t} - \binom{r}{t-1} + - \cdots \geq 0$ $\quad \leftarrow$ same if t even

$$\binom{r-1}{t} + \binom{r-1}{t-1} - \binom{r-1}{t-1} - \binom{r-1}{t-2} + \binom{r-1}{t-2} + \binom{r-1}{t-3} - \cdots = \binom{r-1}{t} \geq 0$$

telescoping sum

$$\binom{r}{j} = \binom{r-1}{j} + \binom{r-1}{j-1}$$

$$\binom{r}{0} = \binom{r-1}{0} + 0$$