HONORS ALGORITHMS

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polynomial $f(x) = a_0 + a_1 x + \dots + a_d x^d$ $\quad a_i \in \mathbb{R}$

$\therefore \deg(f) \le d$

$= d$ if $a_d \ne 0$

zero polynomial: all coeffs zero $\deg(0) =$

$$\boxed{\deg(f \cdot g) = \deg(f) + \deg(g)}$$

if $f = 0$ $\quad \deg(0) = \deg(0) + \deg(g) \Rightarrow \deg(0) = \pm\infty$

$\deg(f+g) \le \max(\deg f, \deg g)$

if $g = -f \ne 0$ $\quad \deg(0) \le \max \deg f$ $\quad \therefore \deg(0) = -\infty$

$$f(x) = a_0 + a_1 x + \cdots + a_d x^d$$
$$g(x) = b_0 + b_1 x + \cdots + b_d x^d$$

$n = d+1$ data for each poly.

Calculate $f \cdot g = c_0 + c_1 x + \cdots + c_{2d} x^{2d}$

$$c_3 = a_0 b_3 + a_1 b_2 + a_2 b_1 + a_3 b_0$$

MODEL: multiplication of reals: unit cost
addition/subtraction 0 cost

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schoolbook algorithm: n^2 multiplications

Karatsuba : $n^{1.59}$ " suffice

Divide and Conquer

$n = 8$

$$f(x) = \underbrace{a_0 + a_1 x + a_2 x^2 + a_3 x^3}_{f_1(x)} + \underbrace{a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7}_{x^4 \cdot f_2(x)}$$

$$f = f_1 + x^{\frac{n}{2}} \cdot f_2 \qquad \deg(f_i) + 1 = \frac{n}{2}$$

$$g = g_1 + x^{\frac{n}{2}} \cdot g_2$$

$$(*) \quad f \cdot g = f_1 \cdot g_1 + x^{\frac{n}{2}} (f_1 g_2 + f_2 g_1) + x^n f_2 g_2$$

$T(n)$: # multipl. required

$$T(n) = 4 \cdot T\left(\frac{n}{2}\right)$$

look for a solution in the form $T(n) = n^{\alpha}$ α constant

WLOG $n = 2^k$

$$n^{\alpha} = 4 \cdot \left(\frac{n}{2}\right)^{\alpha} \qquad 4 = 2^{\alpha} \qquad \alpha = 2$$

bad idea

$$(f_1+f_2)(g_1+g_2) = \underline{f_1 g_1} + f_1 g_2 + f_2 g_1 + \underline{f_2 g_2}$$

$$T(n) = 3 \cdot T\left(\frac{n}{2}\right) \qquad T(n) := n^{\alpha}$$

$$n^{\alpha} = 3 \cdot \left(\frac{n}{2}\right)^{\alpha} \qquad 2^{\alpha} = 3 \qquad \alpha = \log_2 3 < 1.59$$

If n not power of 2

still we can do it at cost $T(n) = O(n^{\alpha})$

implied constant: 4

New model: count multipl + additions/subtractions + bookkeeping

$\boxtimes$ $T(n) \le 3 \cdot T\left(\frac{n}{2}\right) + C \cdot n \xrightarrow{\text{Claim}} T(n) = O(n^{\alpha})$

$\alpha = \log_2 3$

$\boxed{\text{WLOG } n = 2^k}$

Guess fctn $\underline{g(n) \ge 3 \cdot g\left(\frac{n}{2}\right) + Cn}$, $\underline{g(1) \ge T(1)}$ ⊗

METHOD OF REVERSE INEQUALITIES

<u>Claim</u> $\underline{(\forall n)(g(n) \ge T(n))}$

<u>Pf</u> induction on k where $n = 2^k$

$k = 0$ ✓

$k \ge 1$, I.H.: true for $k-1$ $\quad 2^{k-1} = \frac{n}{2}$

$$g(n) \underset{\text{by } \otimes}{\ge} 3 \cdot g\left(\tfrac{n}{2}\right) + Cn \underset{\text{I.H.}}{\ge} 3 \cdot T\left(\tfrac{n}{2}\right) + Cn \underset{\boxtimes}{\ge} T(n)$$

✓

guessed this formula

Need to verify that $g(n) = A n^{\alpha} + B n$

A, B constants TBD

$$g(n) \overset{?}{\geq} 3 g\left(\frac{n}{2}\right) + Cn$$

$$\underline{A n^{\alpha}} + Bn \overset{?}{\geq} \underline{3A\left(\frac{n}{2}\right)^{\alpha}} + 3B\frac{n}{2} + Cn$$

$\div n$

$$B \overset{?}{\geq} \frac{3}{2}B + C$$

$$-\frac{B}{2} \geq C$$

$$B \leq -2C$$

let we choose $\boxed{B = -2C}$

$$g(1) \overset{?}{\geq} T(1) = 1$$

$$A + B \overset{?}{\geq} 1$$

$$A - 2C \overset{?}{\geq} 1$$

$$A \geq 2C + 1$$

$$\boxed{A = 2C + 1}$$

$$g(n) = (2C+1)n^{\alpha} - 2C \cdot n$$

$$< (2C+1)n^{\alpha} = O(n^{\alpha})$$