HONORS ALGORITHMS

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HW $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ $A^n = \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}$ in terms of a familiar sequence

Fibonacci numbers

F_0	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}	...
0	1	1	2	3	5	8	13	21	34	55	...

$F_n = F_{n-1} + F_{n-2}$ $(n \geq 2)$ Fib. recurrence

$F_0 = 0$, $F_1 = 1$ initial values

FACT
$$F_n = \frac{1}{\sqrt{5}}\left(\phi^n - \bar{\phi}^n\right)$$

where $\phi = \frac{1+\sqrt{5}}{2}$ golden ratio ≈ 1.618034

$\bar{\phi} = \frac{1-\sqrt{5}}{2}$ ≈ -0.618034

COR
$$F_n = \left\lfloor \frac{\phi^n}{\sqrt{5}} \right\rceil$$

COR $F_n \sim \frac{\phi^n}{\sqrt{5}}$

$\lfloor x \rfloor$ floor fctn (rounded down value)

$\lceil x \rceil$ ceiling " " up

$\lfloor x \rceil$ nearest integer

$\lfloor \pi \rceil = 3$ $\lfloor -\pi \rceil = -3$

$\lfloor \pi \rfloor = 3$ $\lfloor -\pi \rfloor = -4$

HW $b(F_n)$ # of binary digits of F_n

HW Given $\underline{\underline{n}}$, F_n cannot be computed in poly. time

in binary

<u>DEF</u> an input parameter is TINY if it is given in UNARY

5	101	binary	output: binary
	~~IIII~~	unary	

Example: if n tiny then F_n <u>can</u> be computed in poly time

BON Show: given n, m

$(F_n \bmod m)$ can be computed in poly time

be specific

length of input: $\log n + \log m$

$\log = \log_2$

If $C > 1$ then $x = o(C^x)$ $x \to \infty$

Pf: L'Hôpital

If $C, D > 1$ then $x^D = o(C^x)$

<u>DEF</u> a_n grows <u>exponentially</u> if

$\exists C > 1, c > 0$ s.t.

$$a_n = \Omega(C^{n^c})$$

C^n: simply exponential

e.g. $1.001^{n^{0.001}}$

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e.g. $e^{\sqrt{n}}$ is exp.

HW $\forall C, D > 1, \forall c > 0$

$$x^D = o\left(C^{x^c}\right)$$

e.g. $$x^{100} = o\left(1.001^{x^{0.01}}\right)$$

DO NOT USE L'Hôpital

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bootstrapping

READING: Baron Munchhausen's tales ☺

KNAPSACK PROBLEM

INPUT: $w_1 \dots w_n \in \mathbb{R}$ $w_i > 0$ weights

$v_1 \dots v_n \in \mathbb{R}$ $v_i > 0$ values

$W \in \mathbb{R}$ weight limit

GOAL $\max_{I \subseteq [n]} \left\{ \sum_{i \in I} v_i \;\middle|\; \sum_{i \in I} w_i \leq W \right\}$

2^n trials : trivial algorithm

cost: addition and comparison of reals

bookkeeping

Thm

If $w_i, W \in \mathbb{Z}$ (still unit cost operations)

then one can solve this in $O(nW)$

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poly time for tiny weights

(WLOG $W \leq \sum_{i=1}^{n} w_i$)

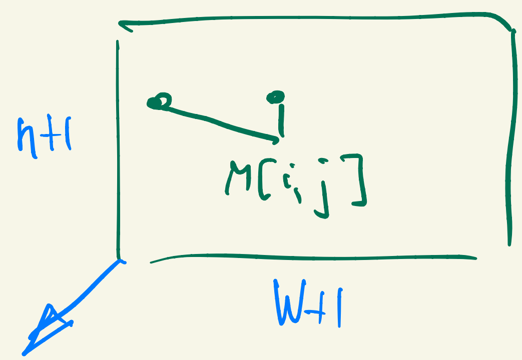
embed the problem into an array of $n \cdot W$ probl.

$$M[i,j] \qquad 0 \le i \le n, \quad 0 \le j \le W$$

$$M[i,j] = \max_{I \subseteq [i]} \left\{ \sum_{t \in I} v_t \;\middle|\; \sum_{t \in I} w_t \le j \right\}$$

brain of the solution

$$M[0,j] = 0$$
$$M[i,0] = 0$$



if $i,j \ge 1$ then

$$M[i,j] = \max \{ M[i-1,j],\; v_i + M[i-1, j-w_i] \}$$

case: item i not included | included

heart of the solution

← DYNAMIC PROGRAMMING

Handout: The Knapsack Problem