PROBLEM SESSION

2024-02-02

S_n : set of permutations of $[n]$

$\sigma : [n] \to [n]$ bijection

S_n : symmetric group of degree n

$A = (a_{ij})_{n \times n}$

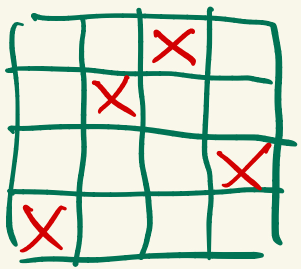
$$\det A = \sum_{\sigma \in S_n} \pm \underbrace{\prod a_{i,\sigma(i)}}_{\text{expansion term}}$$

$\mathrm{sgn}(\sigma)$

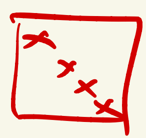
$|S_n| = n!$

expansion term example $a_{13}a_{22}a_{34}a_{41}$

x	1	2	3	4
$\sigma(x)$	3	2	4	1



rook configuration



6.68

$$|\det A| \le \prod_{i=1}^{n} \| a_i \|$$

$\underline{a_i} = (a_{i1} \dots a_{in})$ i^{th} row

$\| a_i \| = \sum_{j=1}^{n} |a_{ij}|$ ℓ_1-norm

$B = (b_{ij})$ $b_{ij} = |a_{ij}|$

$$\|\underline{a_i}\|_1 = \sum_{j=1}^{n} b_{ij}$$

$$\begin{vmatrix} b_{11} & \boxed{b_{1j}} & \cdots & b_{1n} \\ & & \square & \\ & \square & & \end{vmatrix}$$

$$\prod \|\underline{a_i}\|_1 = \sum_{\substack{\sigma \\ \sigma : [n] \to [n]}} \prod_{i=1}^{n} b_{i\sigma(i)} \ge \sum_{\sigma \in S_n} \prod b_{i\sigma(i)}$$

$\ge |\det A|$ by triangle ineq $|\sum c_j| \le \sum |c_j|$

6.74

$$D_5 = \begin{vmatrix} 1 & 1 & & & 0 \\ -1 & 1 & 1 & & \\ & -1 & 1 & 1 & \\ & & -1 & 1 & 1 \\ 0 & & & -1 & 1 \end{vmatrix}$$

tridiagonal matrix

3

expand by first row

$$D_n = D_{n-1} + D_{n-2}$$

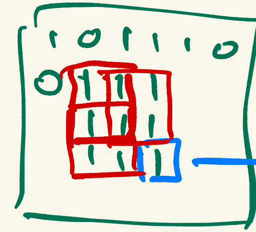
by induction: $\boxed{D_n = F_{n+1}}$

$$D_1 = |1| = 1 = F_2$$

$$D_2 = \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = 2 = F_3$$

6.88 all-ones square

dynamic programming



(i,j)

$m[i,j]$ = max size of an all-ones square of which bottom-right corner is (i,j)

$A = (a_{ij})$

$a_{ij} \in \{0,1\}$

init: $X=0$, **for** $i=1$ **to** n
  for $j=1$ **to** n $m[i,j]=0$

for $i=1$ **to** n
  for $j=1$ **to** n
    if $a_{ij}=1$ **then**
      $m[i,j] = 1 + \min\{m[i-1,j], m[i,j-1], m[i-1,j-1]\}$

for $i=1$ **to** n
  for $j=1$ **to** n
    $X := \max(X, m[i,j])$

7.55 reverse digraph

$A[i]$ list of out-neighbors of vertex i

Need to create $A^{tr}[j]$ list of in-neighbors of j

SINGLE PASS

<u>for</u> $i = 1$ <u>to</u> n (vertices)

  <u>for</u> $j \in Adj[i]$ $(i \to j)$

    <u>add</u> i to $Adj^{tr}[j]$ $(j \xrightarrow{tr} i)$

} monotone lists for G^{tr}

7.56 monotone adjacency lists for G:

$(G^{tr})^{tr}$

7.62 eliminating repetitions:

in monotone list, skip repeated entry

7.68 strong connectivity

DEF digraph G is strongly connected

if $(\forall i,j \in V)$(j is accessible from i)

Lemma Pick vertex s

G is str. conn $\Longleftrightarrow$ (1) $(\forall v \in V)$(v is accessible from s in G)

(2) — " — — " — in G^{tr})

BFS(G)

BFS(G^{tr})

edge-list representation

V - array

E - list

DO convert in lin time between Adj. list and edge-list rep.

do radix sort on edge list

$\Rightarrow$ monotone

6.48 BON

DEF $a_0, a_1, a_2, \ldots$ is a <u>Fibonacci-type</u> sequence if $(\forall n \geq 2)(a_n = a_{n-1} + a_{n-2})$

$(1, g, g^2, \ldots)$ is Fib-type $\iff$

$g = \phi$ or $\bar{\phi}$

$\phi = \frac{1+\sqrt{5}}{2}$ $\bar{\phi} = \frac{1-\sqrt{5}}{2}$

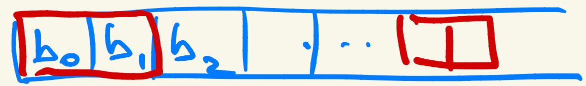
<u>Proof</u>. necessary: $g^2 = g + 1$

Sufficency: $\downarrow$

$\times\, g^{n-2}$ $g^n = g^{n-1} + g^{n-2}$

(b) If (b_n) Fib type then $\exists\, r, s$

$\forall n \qquad b_n = r \cdot \phi^n + s \cdot \bar{\phi}^n$ ←



Lemma If $(c_n), (d_n)$ are Fib. type then

$(\forall r, s)\ (r c_n + s d_n)$ — " —

Lemma $\dim(\text{Fib-type seq.}) = 2$

Pf: basis: $e_1 (1, 0, \dots)$

$e_2 = (0, 1, \dots)$

$(b_0, b_1, \dots)$ — unique comb of e_1, e_2

$\therefore$ $\left.\begin{array}{l}(a_0, a_1, \dots) \\ (a_0', a_1', \dots)\end{array}\right\}$ Fib. type

form a basis of — " — seq. $\iff$

$\begin{vmatrix} a_0 & a_1 \\ a_0' & a_1' \end{vmatrix} \neq 0$ i.e. $a_0 a_1' \neq a_0' a_1$

$\left[\begin{array}{l} 1, \phi, \dots \\ 1, \bar{\phi}, \dots \end{array}\right.$ $\qquad \bar{\phi} \neq \phi$ ✓

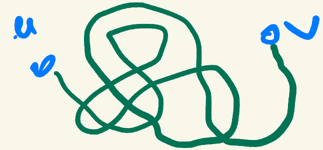
$\therefore$ a basis of space of Fib. type seq.

(c) Binet:

$$r \cdot 1 + s \cdot 1 = 0 = F_0$$
$$r \cdot \phi + s \cdot \bar{\phi} = 1 = F_1$$

9.31 BON reduce walk to path

$W[u_0 \ldots u_m]$



$O(n+m)$

$[3, 1, 5, 7, 1, 7, 4, 5, 1, 7, 5, 7, 8]$

length m

last occurrence of 1

B of length n

1	~~0~~	~~5~~ 7
2	0	
3	1	
4	~~0~~	5
5	~~0~~	~~7~~ ~~1~~ 7
6	0	
7	~~0~~	~~4~~ ~~4~~ ~~5~~ 8
8	0	
9	0	

for $j = 1$ to $m-1$

$B[u_j] := u_{j+1}$

if $x \in \{u_0 \ldots u_m\}$ then $last(x) = \max\{j \mid u_j = x\}$

output $P = [p_0 \cdots p_\ell]$, $x = u_0$

while $x \neq u_m$, add x to P

$x := B[x]$

$\Rightarrow$ alg. terminates

Lemma $last[B[x]] > last[x]$ $\Longrightarrow$ gets to u_m

COROLLARY $last(p_0) < last(p_1) < \ldots$ $\Rightarrow$ no repeats

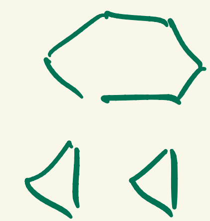
Solution #2

H : vertices & edges of walk

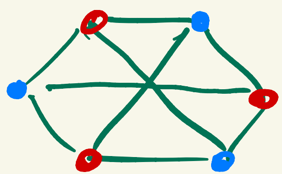
Lemma : $\exists$ path from u_0 to u_m

Pf : shortest walk is a path

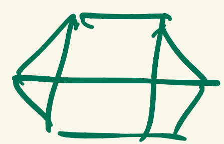
BFS(H) ✓



8.72 $G \not\cong H$ but regular, same deg, same n



G



H

b/c

$H \supseteq K_3$ $\triangle$

$G \not\supseteq K_3$