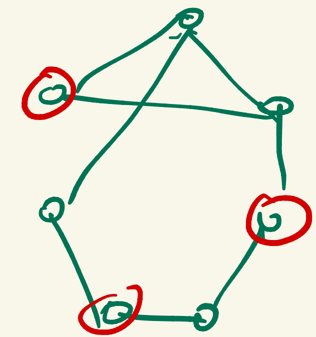
HONORS ALGORITHMS

2024-02-05

Graph (undirected) $G = (V, E)$



<u>DEF</u> $A \subseteq V$ A is <u>independent</u> if there are no edges among A

independence number

$$\alpha(G) = \max\{|A| : A \text{ indep.}\}$$

Same as the <u>clique number</u> of the complement $\overline{G}$

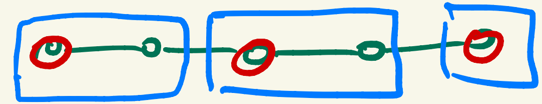
↗ size of largest clique

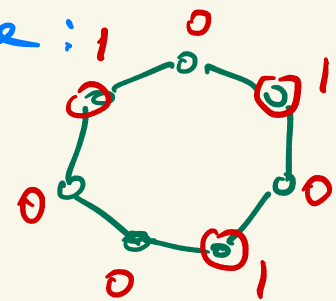
$\alpha(C_n) = \lfloor \frac{n}{2} \rfloor$ cycle of length n

$\alpha(P_n) = \lceil \frac{n}{2} \rceil$ path of length $n-1$

↑

this is maximum by PHP pigeon hole principle:





$V = \{0, 1, \dots, n-1\}$

$f : V \to \{0,1\}$ characteristic fcn of indep set A

$f(i-1) + f(i) \leq 1$, $f(n-1) + f(0) \leq 1$ ← n inequalities

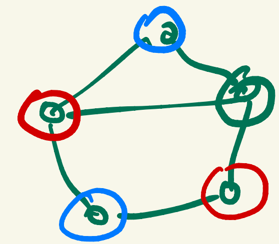
SUM: $2 \cdot \sum f(i) \leq n$

$|A| = \sum f(i) \leq \frac{n}{2}$

legal color of a graph $G = (V, E)$

$f : V \to \{\text{colors}\}$

$i \sim j \implies f(i) \neq f(j)$



$\therefore$ (legal) 3-coloring

this is optimal: $G \supset K_3$ Δ

DEF Chromatic number of G:

min # colors for a legal coloring

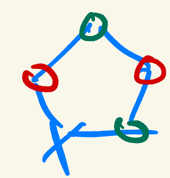
$\chi(G) = ?$

$\chi(K_n) = n$, $\chi(\overline{K_n}) = 1$

DO $\chi(G) = n \iff G \cong K_n$

$\chi(P_n) = 2$
$n \geq 2$

$\chi(C_n) = \begin{cases} 2 & \text{if } n \text{ even} \\ 3 & \text{if } n \text{ odd} \end{cases}$
$n \geq 3$



If $H \subseteq G$ then $\chi(H) \leq \chi(G)$

$\therefore$ If $G \supset K_r$ then $\chi(G) \geq r$

Find G s.t. $G \not\supset K_4$ yet $\chi(G) \geq 4$

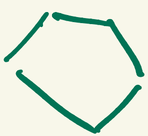
DEF <u>Cone</u> of G is G':

$V(G') = V(G) \cup \{x\}$ ← new vertex

x is adj. to all vertices of G

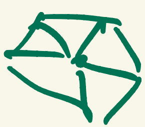
$E(G') = E(G) \cup$ — " —

C_5



Find G: $G \not\supset K_3$ triangle-free

C_5'



DO $\chi(G) \geq 4$

$n = 11$, 5-fold rotational sym

CH $(\forall k)(\exists G)(\not\supset K_3,\ \chi(G) \geq k)$

GREEDY COLORING

$V = [n]$ colors: $\mathbb{N}$

for $i = 1$ **to** n

$c(i) := \min\{ j \mid c(i) = j$ does not conflict with colors $c(\ell),\ \ell < i \}$

DO # colors used by GREEDY $\leq 1 + \Delta(G)$

$\Delta(G)$ — max degree

"GREEDY is terrible"

EX. Find bipartite graph $\chi(G) \leq 2$
= 2-colorable

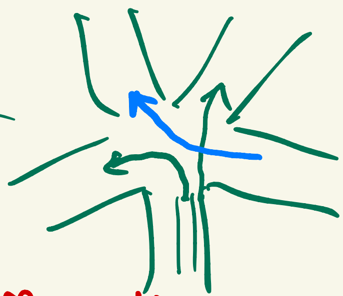
s.t. GREEDY uses $\frac{n}{2}$ colors (assume n even)

EX. $(\forall G)(\exists$ numbering of vertices s.t. GREEDY is optimal$)$

EX. $\alpha(G) \cdot \chi(G) \geq n$

Chromatic number: conflict model

frequency allocation
– radio spectrum management



traffic lights associated with lanes

System of linear equations

n equations in n unknowns

uniquely solvable $\iff$

A is nonsingular

<u>DEF</u> $\det(A) \neq 0$

<u>EX</u> this is equiv. to each of:

$rk(A) = n$

columns of A are li indep

rows " " " "

$\exists A^{-1}$

. . .

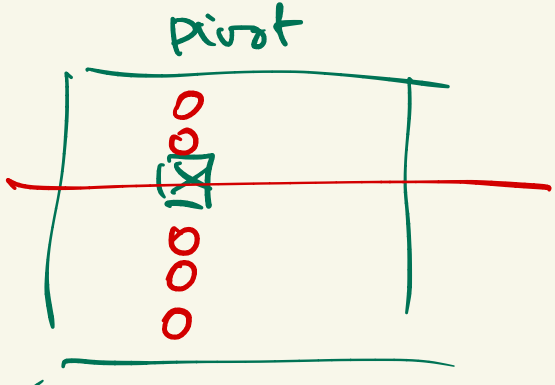
$A\underline{x} = \underline{b}$

A: $n \times n$ matrix

$\underline{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ unknowns

$\underline{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$ RHS

GAUSSIAN ELIMINATION



$O(n^3)$ arithm. operations

Assume $A = (a_{ij})$ $a_{ij} \in \mathbb{Z}$, $b_j \in \mathbb{Z}$

is then Gaussian elim. poly time?

$$\frac{x}{y} - \frac{u}{v} = \frac{xv - yu}{yv}$$

control bit-length ⟶ G.E. is poly. time

Jack Edmonds 1965