HONORS ALGORITHMS

2024-02-09

NEW HANDOUT: WALK-TO-PATH

Link from HW page, 15.10

POLYNOMIAL IDENTITY LEMMA

$f \in F[x_1, \dots x_n]$ $\quad F \cdot$ field

$H \subseteq F$

If $f \not\equiv 0$ (formally zero) (all coeff's zero)

then $\displaystyle P_{a_i \in H}(f(a_1, \dots, a_n) = 0) \leq \frac{d}{|H|}$ ← degree

← finite

Proof: induction on n

$n = 1$ $\quad \displaystyle P_{a \in H}(f(a) = 0) \leq \frac{d}{|H|}$

b/c f has at most d roots in F

3

$\deg f = d$

now assume $n \geq 2$

I.H. Thm true for $n' < n$ variables

$$A = \{ \underline{a} \in H^n \mid f(\underline{a}) = 0 \} \qquad \underline{a} = (a_1 \dots a_n)$$

<u>NTS</u> $\quad |A| \leq d \cdot h^{n-1} = \frac{d}{h} \cdot h^n \qquad h = |H|$

$$f(x_1 \dots x_n) = g_0(\underline{x}) + g_1(\underline{x}) \cdot x_n + \dots + \underline{g_k(\underline{x}) \cdot x_n^k} \qquad \underline{x} = (x_1 \dots x_{n-1})$$

$g_k \neq 0$

$$B = \{ \underline{b} \in H^{n-1} \mid g_k(\underline{b}) = 0 \} \qquad \deg g_k \leq d-k$$

I.H. $\Rightarrow |B| \leq (d-k) \cdot h^{n-2}$

$$f(x_1 \cdots x_n) = g_0(\underline{x}) + g_1(\underline{x})\cdot x_n + \cdots + g_k(\underline{x})\cdot x_n^k$$

$\underline{x} = (x_1 \cdots x_{n-1})$

$g_k \neq 0$

$$B = \{ \underline{b} \in H^{n-1} \mid g_k(\underline{b}) = 0 \}$$

$\deg g_k \leq d-k$

I.H. $\Rightarrow |B| \leq (d-k)\cdot h^{n-2}$ ⊗

If $\underline{b} \notin B$ then $g_k(\underline{b}) \neq 0$

$$C(\underline{b}) = \{ a \in H \mid f(\underline{b}, a) = 0 \}$$

$|C(\underline{b})| \leq k$

$g_k(b) \neq 0 \quad \therefore \quad f(\underline{b}, x_n)$ has $\deg = k$

$$\#\{ (\underline{b}, a) \mid \underline{b} \in H^{n-1} \setminus B \wedge f(\underline{b}, a) = 0 \} \leq h^{n-1}\cdot k$$

$$\#\{ (\underline{b}, a) \mid \underline{b} \in B \wedge \text{—} \text{''} \text{—} \} \leq |B|\cdot h \leq h^{n-1}\cdot(d-k) \quad ⊗$$

$$\therefore \#\{ (\underline{b}, a) \mid (\underline{b}, a) \in H^n,\ f(\underline{b}, a) = 0 \} \leq h^{n-1}\cdot d \quad \checkmark$$

This bound is <u>tight</u> for every field $\mathbb{F}$ and every (n, d, h) s.t. $d \le h \le |\mathbb{F}|$, $\forall H \subseteq \mathbb{F}$, $|H| = h$

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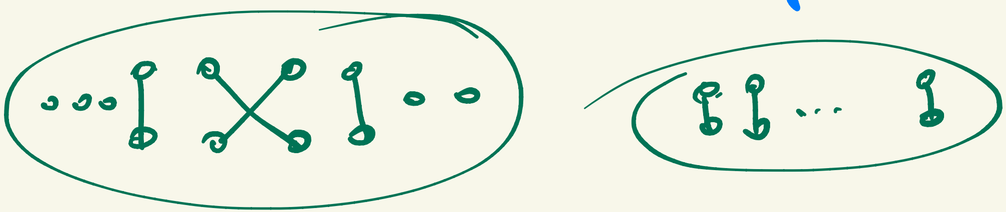
i.e. $\exists f \in \mathbb{F}[x_1, \cdots x_n]$, $\forall H \subseteq \mathbb{F}$, $|H| = h$

s.t. $\displaystyle P_{\underline{a} \in H^n}(f(\underline{a}) = 0) = \frac{d}{h}$

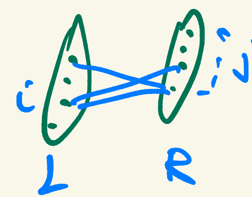
Solution: $\displaystyle f(x_1 \ldots x_n) = \prod_{h \in D} (x_1 - h)$ where $D \subseteq H$, $|D| = d$

Perfect matching in graph G: matching of size $\frac{n}{2}$

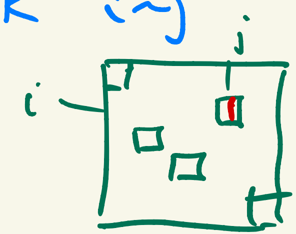


Bipartite graph with $2n$ vertices, n on each side

bipartite adjacency matrix: $n \times n$



$$a_{ij} = \begin{cases} 1 & \text{if } i \in L,\ j \in R \quad i \sim j \\ 0 & \text{o/w} \end{cases}$$



perfect matching: rook configuration

$A = (a_{ij})$

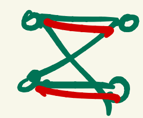
$\boxed{\text{DO}}$ $\exists$ perf matching $\Longleftarrow$ $\det(A) \neq 0$

$\Longrightarrow$ ✗

$\Longleftarrow$

$\Longleftrightarrow$ ✗

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$



C_4

$$A_G = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \mapsto \begin{bmatrix} x_{11} & x_{12} & 0 \\ x_{21} & 0 & x_{23} \\ 0 & 0 & x_{33} \end{bmatrix} = B_G(x_{11} \dots ?)$$

Thm $\exists$ perf matching $\Longleftrightarrow$ $\det B_G \neq 0$

$\Longleftarrow$ ✓

$\Longrightarrow$ no cancellation over any field

$\det B_G \in F[x_1 \cdots x_m]$ deg $= n$ unless $-\infty$

works over $\mathbb{F}_p$ where $p \geq 2n$