HONORS ALGORITHMS

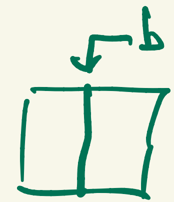
2024-02-12

$$\underset{k \times n}{A}\,\underline{x} = b \qquad \underline{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \qquad \underline{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_k \end{bmatrix}$$

Case : $n \times n$, . $\det A \neq 0$

nonsingular A

CRAMER's rule

$x_i :=$ 

$\det(A)$

$n \in \mathbb{N}$ bit length: $b(n) = \lceil \log_2(n+1) \rceil$ (2

$b(0) := 1$

$$b\left(\frac{r}{s}\right) := b(r) + b(s)$$

$A = (a_{ij})$ $\quad b(A) := \sum_{i,j} b(a_{ij})$

Thm

$$b(\det(A)) \leq b(A)$$

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$D = (d_{ij})$ permanent

DEF

$$\mathrm{per}(D) = \sum_{\sigma \in S_n} \prod d_{i\sigma(i)}$$

L.G. Valiant: per of (0,1)-matrices is #P-complete

"counting"

$c_{ij} = \max(1, b_{ij})$

$A = (a_{ij})$ $n \times n$ $\quad a_{ij} \in \mathbb{Z}$ $\quad B = (\underbrace{|a_{ij}|}_{b_{ij}})$ $\quad C := (c_{ij})$

$$|\det A| \le \mathrm{per}\, B \le \mathrm{per}\, C$$

triangle ineq. $\quad b(A) = b(B) = b(C)$

suffices
<u>NTS</u> $b(\mathrm{per}\, C) \le b(C) \le \sum \lceil \log_2(1+c_{ij}) \rceil$ $\Big($ 4

$C = (c_{ij})$ $\quad$ $\underline{c_{ij} \ge 1}$

$$\mathrm{per}\, C \le \prod \|c_i\|_1 \qquad \|c_i\|_1 = \sum_{j=1}^{n} |c_{ij}|$$

ℓ^1- norm

<u>LEMMA</u> $x_i \in \mathbb{R}$ $\quad x_i \ge 0$

$$\Longrightarrow \prod_{i=1}^{n} (1+x_i) \ge 1 + \sum x_i$$
$$\ge 1 + \prod x_i$$

LEMMA $x_i \in \mathbb{R}$ $x_i \geq 0$

$$\Longrightarrow \prod_{i=1}^{n} (1+x_i) \geq 1 + \sum x_i$$
$$\geq 1 + \prod x_i$$

NTS $\lceil \log_2 (1 + \operatorname{per} C) \rceil \overset{?}{\leq} \sum_{ij} \lceil \log_2 (1 + c_{ij}) \rceil$

WILL SHOW $1 + \operatorname{per} C \overset{?}{\leq} \prod_i \prod_j (1 + c_{ij})$

$$LHS \leq 1 + \prod_{i=1}^{n} \|c_i\|_1 \leq \prod_{i=1}^{n} (1 + \|c_i\|)$$

$$1 + \sum_j c_{ij} \leq \prod_j (1 + c_{ij})$$

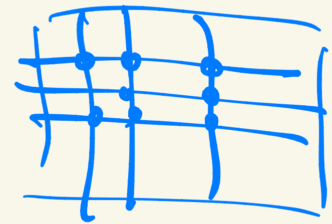
✓

EDMONDS

GAUSSIAN ELIMINATION,
row-operations only

$\Longrightarrow$ for every intermediate matrix

" entry : $\dfrac{D_1}{D_2}$



$\therefore$ $\rightarrow$ in poly. time

KARATSUBA

- multiplication of polynomials | # scalar operations
- mult of integers | $n^{\log_2 3}$

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poly multipl.: $O(n \log n)$

Fast Fourier Transform (over $\mathbb{C}$)

COOLEY-TUKEY 1950s

number multipl SCHÖNHAGE-STRASSEN 1971

$n \log n \cdot \log\log n$

MARTIN FÜRER 2007 $n \cdot \log n \cdot 2^{O(\log^* n)}$

$$\mathrm{tower}(2,n) = 2^{2^{\cdot^{\cdot^{\cdot^{2}}}}} \Big\} n \text{ times}$$

$$\log^* n = \min \{ k \mid \mathrm{tower}(2,k) \geq n \}$$

DFT discrete Fourier transform

$$(a_0, \dots, a_{n-1}) \xmapsto[\text{lin. trf.}]{} (b_0, \dots, b_{n-1})$$

$$f(x) = a_0 + a_1 x + \dots + a_{n-1} x^{n-1}$$

$$\boxed{b_j = f(\omega^j)}$$

$$\underline{b} = F \cdot \underline{a} \qquad F = (\omega^{ij})_{n \times n}$$

$$\underline{b} = \hat{\underline{a}}$$

n^{th} roots of unity

$$1, \omega, \omega^2, \dots, \omega^{n-1}$$

$$\omega = \cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n}$$

$$\overset{a_i}{f(x)} \cdot \overset{c_i}{g(x)} = \underbrace{a_0 c_0}_{\sum a_k c_{-k}} + \underbrace{(a_0 c_1 + a_1 c_0)}_{\sum a_{k+1} c_{-k}} x + \underbrace{(a_0 c_2 + a_1 c_1 + a_2 c_0)}_{\cdots} x^2 + \cdots$$

wraparound

~~convolution~~

$$\widehat{\operatorname{conv}(\underline{a}, \underline{c})} = \hat{a} \cdot \hat{c} = (\hat{a}_0 \hat{c}_0, \hat{a}_1 \hat{c}_1, \ldots, \hat{a}_{n-1} \hat{c}_{n-1})$$

$$(\underline{a}, \underline{c}) \xmapsto{F} (\hat{\underline{a}}, \hat{\underline{c}}) \longmapsto (\widehat{ac}) \xmapsto{F^{-1}} \operatorname{conv}(\underline{a}, \underline{c})$$

$\boxed{\text{DO}}$ $F \cdot F^* = nI$ $\quad F^{-1} = \frac{1}{n} F^*$

$\llcorner$ conjugate - transpose